Propagation and blocking in periodically hostile environments

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Abstract

We study the persistence and propagation (or blocking) phenomena for a species in periodically hostile environments. The problem is described by a reaction-diffusion equation with zero Dirichlet boundary condition. We first derive the existence of a minimal nonnegative nontrivial stationary solution and study the large-time behavior of the solution of the initial boundary value problem. To the main goal, we then study a sequence of approximated problems in the whole space with reaction terms which are with very negative growth rates outside the domain under investigation. Finally, for a given unit vector, by using the information of the minimal speeds of approximated problems, we provide a simple geometric condition for the blocking of propagation and we derive the asymptotic behavior of the approximated pulsating travelling fronts. Moreover, for the case of constant diffusion matrix, we provide two conditions for which the limit of approximated minimal speeds is positive.

1 Introduction and main results

This paper is concerned with persistence and propagation phenomena for reaction-diffusion equations of the type

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u) \tag{1.1}$$

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in $\mathbb{R}^N$ or in some unbounded open subsets Ω of $\mathbb{R}^N$ with zero Dirichlet boundary condition on $\partial\Omega$. Equations of the type (1.1) arise especially in population dynamics and ecological models (see e.g. [25, 33, 37]), where the nonnegative quantity u typically stands for the concentration of a species.

Let us start with the case of the whole space $\mathbb{R}^N$. The symmetric matrix field $x \mapsto A(x) = (A_{ij}(x))_{1\le i,j\le N}$ is assumed to be of class $C^{1,\alpha}(\mathbb{R}^N)$ with $\alpha > 0$ and uniformly positive definite: that is, there exists a positive constant $\beta > 0$ such that

$$\forall\, x \in \mathbb{R}^N, \quad \forall\, \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N, \quad A\xi \cdot \xi := \sum_{1\le i,j\le N} A_{ij}(x)\xi_i\xi_j \ge \beta\, |\xi|^2, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. We set $\mathbb{R}_+ = [0, +\infty)$. The nonlinear reaction term $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$, $(x, u) \mapsto f(x, u)$ is assumed to be continuous, of class $C^{0,\alpha}$ with respect to x locally uniformly in $u \in \mathbb{R}_+$, of class C^1 with respect to u, and $\frac{\partial f}{\partial u}(\cdot, 0)$ is of class $C^{0,\alpha}(\mathbb{R}^N)$. Furthermore, we assume that

$$\begin{cases} f(x, 0) = 0 \text{ for all } x \in \mathbb{R}^N, \\ \text{there exists } M > 0 \text{ such that } f(x, M) \le 0 \text{ for all } x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

The functions A_{ij} (for all $1 \le i, j \le N$) and $f(\cdot, u)$ (for all $u \in \mathbb{R}_+$) are assumed to be periodic in $\mathbb{R}^N$. Hereafter a function w is called periodic in $\mathbb{R}^N$ if it satisfies

$$w(\cdot + k) = w(\cdot) \;\text{ for all } k \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z},$$

where $L_1, \cdots, L_N$ are some positive real numbers, which are fixed throughout this paper.

If f fulfills the additional Fisher-KPP (for Kolmogorov, Petrovsky and Piskunov) [14, 20] assumption

$$\forall\, x \in \mathbb{R}^N, \quad u \mapsto g(x, u) = \frac{f(x, u)}{u} \text{ is decreasing with respect to } u > 0, \tag{1.4}$$

then the large-time behavior of the solutions of the Cauchy problem

$$\begin{cases} u_t - \nabla\cdot(A(x)\nabla u) &=& f(x, u), \quad t > 0,\ x \in \mathbb{R}^N, \\ u(0, x) &=& u_0(x), \qquad x \in \mathbb{R}^N \end{cases} \tag{1.5}$$

is directly related to the sign of the principal periodic eigenvalue λ_1 of the linearized operator at 0 (see [5]). This eigenvalue λ_1 is characterized by the existence of a (unique up to multiplication) periodic function $\varphi \in C^{2,\alpha}(\mathbb{R}^N)$, which satisfies

$$\begin{cases} -\nabla\cdot(A(x)\nabla\varphi) - \zeta(x)\varphi &=& \lambda_1\,\varphi \quad \text{in } \mathbb{R}^N, \\ \varphi &>& 0 \qquad \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $\zeta(x) = \frac{\partial f}{\partial u}(x, 0)$ for all $x \in \mathbb{R}^N$. The precise statement of what is known under the additional assumption (1.4) will be recalled just after Proposition 1.1 below.

Our first result, which is a preliminary step before the main purpose of the paper devoted to propagation phenomena in environments with hostile boundaries, is actually concerned with the existence of a minimal positive stationary solution p for problem (1.5) and with the large time behavior of the solutions u of (1.5), when f fulfills the assumption (1.3) alone.

Proposition 1.1 *Assume that* $\lambda_1 < 0$ *and* (1.3)*. Then there is a minimal periodic solution* $p(x)$ *of*

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) &= f(x, p(x)) \quad \text{in } \mathbb{R}^N, \\ p &> 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

in the sense that, for any solution q *of* (1.7)*, there holds* $q \geq p$ *in* $\mathbb{R}^N$*. Furthermore,* $p \leq M$ *in* $\mathbb{R}^N$ *and, if* $u_0 : \mathbb{R}^N \to [0, M]$ *is uniformly continuous and not identically* 0*, then the solution* $u(t, x)$ *of the Cauchy problem* (1.5) *is such that*

$$\liminf_{t \to +\infty} u(t, x) \geq p(x) \quad \text{locally uniformly with respect to } x \in \mathbb{R}^N.$$

If one further assumes that $u_0 \leq p$ *in* $\mathbb{R}^N$*, then* $u(t, x) \to p(x)$ *as* $t \to +\infty$ *locally uniformly with respect to* $x \in \mathbb{R}^N$*.*

It is obvious to see that the solution p of (1.7) is not unique in general. Choose for instance $A(x) = I_N$ (the identity matrix) and $f(x, u) = \sin(u)$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}_+$: the function f satisfies (1.3) with $M = \pi$, $\lambda_1 = -1 < 0$, but any constant function $p(x) = m\pi$ with $m \in \mathbb{N}\backslash\{0\}$ solves (1.7). On the other hand, if, in addition to (1.3), the function f satisfies the assumption (1.4), then the solution p of (1.7) is unique, see [5]. In particular, all solutions of (1.7) are necessarily periodic. Notice that, in the general case of assumption (1.3) alone, Proposition 1.1 still states the existence of a minimal periodic solution p of (1.7) in the class of all positive solutions q, which are not a priori assumed to be periodic. It is also known that, under hypotheses (1.3) and (1.4), the condition $\lambda_1 < 0$ of the unstability of 0 is a necessary condition for the existence of the solution p of (1.7) as well: if $\lambda_1 \geq 0$, then all bounded solutions u of (1.5) converge to 0 as $t \to +\infty$ uniformly in $\mathbb{R}^N$, see [5]. On the other hand, under the assumptions (1.3), (1.4) and $\lambda_1 < 0$, for any non-zero bounded uniformly continuous $u_0 : \mathbb{R}^N \to \mathbb{R}_+$, there holds $u(t, x) \to p(x)$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$ (see [5, 15, 35]). We also refer to [5, 10, 11] for related results in the case of bounded domains with Dirichlet or Neumann boundary conditions, and to [4, 7] for results with KPP nonlinearities and periodic or non-periodic coefficients in $\mathbb{R}^N$. Lastly, it is worth noticing that Proposition 1.1 and the aforementioned convergence results are different from what happens with other types of nonlinearities f, like combustion, bistable or even monostable nonlinearities which are degenerate at 0: in these cases, the large-time behavior of the solutions u of (1.5) strongly depends on some threshold parameters related to the size and/or the amplitude of u_0 (see e.g. [1, 12, 30, 31, 38]).

Remark 1.1 The assumption that u_0 ranges in the interval $[0, M]$ is made to guarantee the global existence and boundedness (from below by 0 and from above by M) of the solutions u of the Cauchy problem (1.5). If f fulfills the KPP assumption (1.4) together with (1.3), or if $f(x, s) \leq 0$ for all $(x, s) \in \mathbb{R}^N \times [M, +\infty)$, then it follows that the solution u exists for all $t \geq 0$ and is globally bounded from below by 0 and from above by $\max\big(M, \|u_0\|_{L^\infty(\mathbb{R}^N)}\big)$, as long as u_0 is nonnegative and bounded. The same comment also holds for the Cauchy problem (1.14) below with zero Dirichlet boundary condition on $\partial\Omega$.

As a matter of fact, in Proposition 1.1, the negativity of λ_1 immediately implies that the positive periodic functions $\varepsilon\,\varphi$ are subsolutions of (1.5) for $\varepsilon > 0$ small enough, where φ is a solution of (1.6). It then follows from the above proposition and the results of Weinberger [35] that, for each unit vector e of $\mathbb{R}^N$, there is a positive real number $c^*(e) > 0$ (minimal speed) such that the following holds: for each $c \geq c^*(e)$, there is a pulsating travelling front

$$u(t,x) = \phi(x\cdot e - ct, x)$$

solving (1.5) and connecting 0 to p, that is, the function $\phi : \mathbb{R}\times\mathbb{R}^N \to [0, M],\ (s,x) \mapsto \phi(s,x)$ is periodic in x, decreasing in s, and it satisfies $\phi(-\infty, x) = p(x)$ and $\phi(+\infty, x) = 0$ for all $x \in \mathbb{R}^N$. Furthermore, such pulsating travelling fronts do not exist for any $c < c^*(e)$. We also refer to [26, 28, 29, 36, 37] for other results about pulsating travelling fronts in the whole space $\mathbb{R}^N$, including other types of nonlinearities and the case of time-periodic media.

Now, based on the previous results in $\mathbb{R}^N$, we turn our attention to the main concern of this paper, namely the case when there are hostile periodic patches in the domain under consideration. We deal with persistence and propagation phenomena for reaction-diffusion equations of the type

$$\begin{cases} u_t - \nabla\cdot(A(x)\nabla u) &= F(x,u), \quad x \in \overline{\Omega}, \\ u(t,x) &= 0, \qquad\qquad x \in \partial\Omega, \end{cases} \tag{1.8}$$

in an unbounded open set $\Omega \subset \mathbb{R}^N$ which is assumed to be of class $C^{2,\alpha}$ (with $\alpha > 0$) and periodic. The periodicity means that $\Omega = \Omega + k$ for all $k \in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$. Furthermore, the fields $A(x)$ and $F(x,u)$ are assumed to be periodic with respect to x in $\overline{\Omega}$, to have the same smoothness as before and to fulfill (1.2) and (1.3) above, where $x \in \mathbb{R}^N$ is now replaced with $x \in \overline{\Omega}$. In particular, assumption (1.3) is now replaced with

$$\begin{cases} F(x,0) = 0 \text{ for all } x \in \overline{\Omega}, \\ \text{there exists } M > 0 \text{ such that } F(x,M) \leq 0 \text{ for all } x \in \overline{\Omega}. \end{cases} \tag{1.9}$$

Throughout the paper, we denote

$$C = \overline{\Omega} \cap \big([0,L_1]\times\cdots\times[0,L_N]\big)$$

the cell of periodicity of $\overline{\Omega}$. The zero Dirichlet boundary condition imposed on $\partial\Omega$ means that the boundary is lethal for the species. Note that the unbounded periodic open set Ω is not a priori assumed to be connected. The reason for that will become clear later, once the approximation procedure (1.16) below has been introduced. However, due to the global smoothness of $\partial\Omega$, the set Ω has only a finite number of connected components relatively to the lattice $L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$. That is, there is a finite number of connected components $\omega_1,\ldots,\omega_m$ of Ω such that $\omega_i \cap (\omega_j + k) = \emptyset$ for all $1 \leq i \neq j \leq m$ and for all $k \in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$, and

$$\Omega = \bigcup_{1\leq i\leq m} \Omega_i, \quad \text{where } \Omega_i = \bigcup_{k\in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}} \omega_i + k. \tag{1.10}$$

The sets ω_i are not uniquely defined, but the sets Ω_i are unique (up to permutation), periodic, and $\Omega_i \cap \Omega_j = \emptyset$ for all $1 \le i \neq j \le m$.

In the case of no-flux boundary conditions $\nu(x) \cdot (A(x)\nabla u(t,x)) = 0$ on $\partial\Omega$ when Ω is connected, much work have been devoted in the recent years to the study of propagation of pulsating fronts $u(t,x) = \phi(x \cdot e - ct, x)$, where $\phi(s,\cdot)$ is periodic for all $s \in \mathbb{R}$ and e is any unit vector, for various types of nonlinearities F, in straight infinite cylinders [8, 31] or in periodic domains [3, 16, 17, 22, 35]. In the case of KPP nonlinearities F, further properties of the minimal propagation speeds can be found in [6, 13, 18, 19, 21, 27, 32, 39].

In this paper, we consider a larger class of reaction terms F, together with zero Dirichlet boundary condition. Let us first mention that, under the assumption that the equation (1.8) is invariant in the direction x_1 and under appropriate conditions on F, classical travelling fronts

$$u(t,x) = \phi(x_1 - ct, x_2, \ldots, x_n)$$

in straight infinite cylinders (in the x_1-direction) with zero Dirichlet boundary condition are known to exist (see [24, 34], including the case of some systems of equations). In this case, the profiles ϕ of these travelling fronts solve elliptic equations or systems. For problem (1.8) in periodic domains, the reduction to elliptic equations does not hold anymore since the equation is not assumed to be invariant in any direction. Recently, existence results for problems of the type (1.8) in connected two-dimensional periodically oscillating infinite cylinders with homogeneous isotropic diffusion ($A(x) = I_2$) and KPP nonlinearities satisfying (1.4) have been established, see [23]. In the present paper, the set Ω is periodic in all variables $x_1, \ldots, x_N$ and the direction of propagation may be any unit vector e of $\mathbb{R}^N$. Actually, one of the novelties of this paper with respect to the previous literature is that the nature of propagation vs. blocking strongly depends on the direction e and on geometrical properties of the set Ω itself.

Let $\lambda_{1,D}$ denote the principal periodic eigenvalue of the linearized equation at 0 in $\overline{\Omega}$ with zero Dirichlet boundary condition. That is, there exists a function $\varphi \in C^{2,\alpha}(\overline{\Omega})$, which is periodic in $\overline{\Omega}$ and satisfies

$$\begin{cases} -\nabla \cdot (A(x)\nabla\varphi) - \zeta(x)\varphi &= \lambda_{1,D}\,\varphi \quad \text{in } \overline{\Omega}, \\ \varphi &= 0 \quad \text{on } \partial\Omega, \\ \varphi &\geq 0 \quad \text{in } \overline{\Omega}, \\ \max_{\overline{\Omega}} \varphi &> 0, \end{cases} \tag{1.11}$$

where $\zeta(x) = \frac{\partial F}{\partial u}(x,0)$ for all $x \in \overline{\Omega}$. If Ω is connected, then $\varphi > 0$ in Ω and φ is unique up to multiplication. Otherwise, in the general case, the function φ is unique up to multiplication in each set Ω_i on which it is positive. More precisely, φ can be chosen to be positive on the (largest possible) set $\widetilde{\Omega} = \bigcup_{i \in I_{min}} \Omega_i$, where I_{min} denotes the set of indices $i \in \{1, \ldots, m\}$ for which the principal periodic eigenvalue $\lambda_{1,\Omega_i,D}$ of the operator $-\nabla \cdot (A(x)\nabla) - \zeta(x)$ in Ω_i with zero Dirichlet boundary condition on $\partial\Omega_i$ is equal to $\lambda_{1,D}$. That is,

$$\lambda_{1,D} = \min_{1 \le j \le m} \lambda_{1,\Omega_j,D} = \lambda_{1,\Omega_i,D} \text{ for all } i \in I_{min}.$$

The following theorem, which is analogue to Proposition 1.1, is concerned with the existence of a minimal nonnegative and non-trivial stationary solution of (1.8) in $\overline{\Omega}$ and the large-time behavior of the solutions of the associated initial boundary value problem, under the assumption that the steady state 0 of (1.8) is linearly strictly unstable. To do so, we introduce the set

$$I_- = \Big\{ i \in \{1, \ldots, m\}, \quad \lambda_{1,\Omega_i,D} < 0 \Big\}. \tag{1.12}$$

Theorem 1.2 *Assume that $\lambda_{1,D} < 0$, that is $I_- \neq \emptyset$. Then there exists a minimal stationary periodic solution $p(x)$ of*

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) &= F(x, p(x)) & \text{in } \overline{\Omega}, \\ p &= 0 & \text{in } \partial\Omega \cup \bigcup_{i \notin I_-} \Omega_i, \\ p &> 0 & \text{in } \bigcup_{i \in I_-} \Omega_i, \end{cases} \tag{1.13}$$

in the sense that any bounded solution q of (1.13) *satisfies $q \geq p$ in $\overline{\Omega}$. Moreover, for any uniformly continuous function $u_0 : \overline{\Omega} \to [0, M]$ which is not identically* 0*, the solution $u(t, x)$ of the initial boundary value problem*

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) &= F(x, u), & t > 0, \ x \in \overline{\Omega}, \\ u(t, x) &= 0, & t > 0, \ x \in \partial\Omega, \\ u(0, x) &= u_0(x), & x \in \Omega \end{cases} \tag{1.14}$$

is such that

$$\liminf_{t \to +\infty} u(t, x) \geq p(x) \tag{1.15}$$

locally uniformly with respect to the points $x \in \overline{\Omega}$ whose connected components intersect the support of u_0. If one further assumes that $u_0 \leq p$ in $\overline{\Omega}$, then $u(t, x) \to p(x)$ as $t \to +\infty$ in the same sense as above.

As already emphasized, the periodic open set Ω is not assumed to be connected, this is why the lower bound (1.15) or the convergence of $u(t, x)$ to $p(x)$ at large time can only hold in the (open) connected components $\mathcal{C}$ of the intersection of Ω with the support of u_0 (outside these components, the solution $u(t, x)$ stays 0 for all times $t \geq 0$). If such a connected component $\mathcal{C}$ is included in a set Ω_i with $i \in I_-$, then Theorem 1.2 implies that $u(t, x)$ is separated away from 0 at large time, locally uniformly in $\mathcal{C}$. However, (1.15) does not say anything about the behavior of $u(t, x)$ when $x \in \bigcup_{i \notin I_-} \overline{\Omega_i}$ ($p(x) = 0$ there). Actually, for each Ω_i with $i \notin I_-$, one has $\lambda_{1,\Omega_i,D} \geq 0$ and if F satisfies the additional assumption (1.4) in Ω_i, then $u(t, x) \to 0$ as $t \to +\infty$ uniformly in $x \in \overline{\Omega_i}$, as follows from the same ideas as in [5].

The remaining part of this paper is concerned with the existence of pulsating fronts and the possibility of blocking phenomena for problem (1.8) with zero Dirichlet boundary condition. The strategy, which is one of the main interests of the paper, consists in approximating the Dirichlet condition on $\partial\Omega$ (and even in $\mathbb{R}^N \backslash \Omega$) by reaction terms with very negative growth rates in $\mathbb{R}^N \backslash \overline{\Omega}$, using the previous results and then passing to the singular limit in the stationary solutions and in the pulsating travelling fronts as the growth rates converge to $-\infty$ in $\mathbb{R}^N\backslash\overline{\Omega}$. This means that the quantity u lives in the whole space $\mathbb{R}^N$, but the space contains very bad regions. We will see that the location of the good vs. bad regions plays a crucial role in the dynamical behavior of the solutions.

For this, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions defined in $\mathbb{R}^N\times\mathbb{R}_+$ such that each function $f_n:(x,u)\mapsto f_n(x,u)$ is continuous, periodic with respect to $x\in\mathbb{R}^N$, of class $C^{0,\alpha}$ with respect to $x\in\mathbb{R}^N$ locally uniformly in $u\in\mathbb{R}_+$, of class C^1 with respect to u with $\frac{\partial f_n}{\partial u}(\cdot,0)\in C^{0,\alpha}(\mathbb{R}^N)$, and it satisfies (1.3). Here we define $\mathbb{N}$ to be the set of all nonnegative integers. Furthermore, we assume that

$$\begin{cases} f_n(x,u)=F(x,u) & \text{for all } (x,u)\in\overline{\Omega}\times\mathbb{R}_+ \text{ and } n\in\mathbb{N},\\ (f_n(x,u))_{n\in\mathbb{N}} \text{ is nonincreasing} & \text{for all } (x,u)\in\overline{\Omega}\times\mathbb{R}_+,\\ g_n(x,u)\to-\infty \text{ as } n\to+\infty & \text{locally uniformly in } (x,u)\in(\mathbb{R}^N\backslash\overline{\Omega})\times\mathbb{R}_+, \end{cases} \tag{1.16}$$

where

$$g_n(x,u)=\begin{cases} \dfrac{f_n(x,u)}{u} & \text{if } u>0,\\ \dfrac{\partial f_n}{\partial u}(x,0)=:\zeta_n(x) & \text{if } u=0. \end{cases}$$

The last condition means that the death rate in the region $\mathbb{R}^N\backslash\overline{\Omega}$ is very high, namely this region becomes more and more unfavorable for the species as n becomes larger and larger.

Typical examples of such functions f_n satisfying (1.9) and (1.16) are

$$f_n(x,u)=\rho_n(x)\,u+\widetilde{f}(u),$$

where the function $\widetilde{f}:\mathbb{R}_+\to\mathbb{R}$ is of class C^1 and satisfies $\widetilde{f}(0)=0$, $\widetilde{f}(M)\le 0$, and the functions $\rho_n:\mathbb{R}^N\to\mathbb{R}$ are periodic, nonpositive, of class $C^{0,\alpha}(\mathbb{R}^N)$, nonincreasing with respect to n, independent of n in Ω, and $\rho_n\to-\infty$ as $n\to+\infty$ locally uniformly in $\mathbb{R}^N\backslash\overline{\Omega}$.

For every $n\in\mathbb{N}$, let $\lambda_{1,n}$ denote the principal periodic eigenvalue of the linearized operator at 0 in $\mathbb{R}^N$. That is, there exists a (unique up to multiplication) periodic function φ_n of class $C^{2,\alpha}(\mathbb{R}^N)$, which satisfies

$$\begin{cases} -\nabla\cdot(A(x)\nabla\varphi_n)-\zeta_n(x)\varphi_n &=& \lambda_{1,n}\,\varphi_n & \text{in } \mathbb{R}^N,\\ \varphi_n &>& 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.17}$$

We first establish the relationship between the principal eigenvalues $\lambda_{1,n}$ of (1.17) and the principal eigenvalue $\lambda_{1,D}$ of (1.11), as well as the convergence of the minimal solutions p_n of (1.7) with nonlinearities f_n to the minimal solution p of (1.13), when $\lambda_{1,D}<0$.

Theorem 1.3 *Under the above notation, the sequence $(\lambda_{1,n})_{n\in\mathbb{N}}$ is nondecreasing and there holds $\lambda_{1,n}\to\lambda_{1,D}$ as $n\to+\infty$. Furthermore, if $\lambda_{1,D}<0$, then the sequence $(p_n)_{n\in\mathbb{N}}$ of minimal solutions of* (1.7) *with nonlinearities f_n is nonincreasing and*

$$p_n(x)\to p_\infty(x) \text{ as } n\to+\infty \text{ for all } x\in\mathbb{R}^N,$$

where, up to a negligible set, p_∞ is nonnegative, periodic in $\mathbb{R}^N$, $p_\infty = 0$ in $\mathbb{R}^N\backslash\Omega$, the restriction of p_∞ on $\overline{\Omega}$ is of class $C^{2,\alpha}(\overline{\Omega})$ and solves

$$\begin{cases} -\nabla\cdot(A(x)\nabla p_\infty) = F(x,p_\infty) & \text{in } \overline{\Omega}, \\ p_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

Lastly, $p_\infty \geq p$ in $\overline{\Omega}$, where p is given in Theorem 1.2.

We point out that, in general, the function p_∞ is not identically equal to the solution p of (1.13) in $\overline{\Omega}$. However, it is well equal to p in $\overline{\Omega}$ if F fulfills (1.4) in Ω. We refer to Remark 3.1 for more details.

The last result is concerned with the asymptotic behavior as $n\to+\infty$ of the pulsating travelling fronts of the type $\phi_n(x\cdot e-ct,x)$ connecting 0 to p_n (for problem (1.1) in $\mathbb{R}^N$ with nonlinearities f_n) and of their minimal speeds $c_n^*(e)>0$ in any direction e (when $\lambda_{1,n}<0$). The limit shall depend strongly on the direction e and blocking phenomena may occur in general.

Theorem 1.4 *Assume that $\lambda_{1,D}<0$ and let e be any given unit vector of $\mathbb{R}^N$.*

a) The sequence $(c_n^*(e))_{n\in\mathbb{N}}$ is nonincreasing with limit $c^*(e)\geq 0$. If all connected components $\mathcal{C}$ of Ω are bounded in the direction e in the sense that

$$\sup_{x\in\mathcal{C}} |x\cdot e| < +\infty, \tag{1.19}$$

then $c^*(e)=0$.

b) For any $c\geq c^*(e)$ with $c>0$ and for any sequence $(c_n)_{n\in\mathbb{N}}$ such that $c_n\to c$ as $n\to+\infty$ and $c_n\geq c_n^*(e)$, the pulsating travelling fronts $u_n(t,x)=\phi_n(x\cdot e-c_nt,x)$ for (1.1) in $\mathbb{R}^N$ with nonlinearity f_n satisfy

$$u_n(t,x)\to\begin{cases} u(t,x) & \text{in } C^1_t \text{ and } C^2_x \text{ locally in } \mathbb{R}\times\Omega, \\ 0 & \text{in } L^1_{loc}(\mathbb{R}\times(\mathbb{R}^N\backslash\Omega)) \end{cases}$$

up to extraction of a subsequence, where $u(t,x)=\phi(x\cdot e-ct,x)$ is a classical solution of (1.8) with $u_t\geq 0$ in $\mathbb{R}\times\overline{\Omega}$ and $\phi(s,\cdot)$ is periodic in $\overline{\Omega}$ for all $s\in\mathbb{R}$. Moreover, for any given $i\in I_-$, one can shift in time the functions u_n so that $u(-\infty,\cdot)=0$ and $u(+\infty,\cdot)>0$ in Ω_i.

c) Assume here that A is constant. If there exist a unit vector $e'\neq\pm e$ and two real numbers $a<b$ such that

$$\Omega\supset\{x\in\mathbb{R}^N,\ a<x\cdot e'<b\}, \tag{1.20}$$

then $c^*(e)>0$. If there exist a unit vector e', a point $x_0\in\mathbb{R}^N$ and a real number $r>0$ such that e' is an eigenvector of A with $e'\cdot e\neq 0$, and

$$\Omega\supset\{x\in\mathbb{R}^N,\ d(x,x_0+\mathbb{R}e')<r\}, \tag{1.21}$$

where d denotes the Euclidean distance, then $c^*(e)>0$.

Theorem 1.4 provides a simple geometrical condition for the blocking of propagation, in a given direction e, in the presence of hostile periodic patches (by blocking, we mean that $c_n^*(e) \to 0$ as $n \to +\infty$). Consequently, some quantitative estimates of the spreading speeds of the solutions u of the Cauchy problems (1.5) with nonlinearities f_n can be derived. Indeed, for any compactly supported function $u_0 \not\equiv 0$, the solution u of (1.5) with nonlinearity f_n spreads in the direction e with the spreading speed

$$w_n^*(e) = \min_{\xi \in \mathbb{S}^{N-1},\, \xi \cdot e > 0} \frac{c_n^*(e)}{\xi \cdot e},$$

in the sense that $\liminf_{t\to+\infty} u(t, c\,t\,e + x) \geq p_n(x)$ locally uniformly in x if $0 \leq c < w_n^*(e)$, whereas $\lim_{t\to+\infty} u(t, c\,t\,e + x) = 0$ locally uniformly in x if $c > w_n^*(e)$ (see [4, 15, 35]). In particular, $0 < w_n^*(e) \leq c_n^*(e)$. Hence, under condition (1.19), $c^*(e) = c^*(-e) = 0$ and the solution u of (1.5) with nonlinearity f_n spreads as slowly as wanted in the directions $\pm e$ when n is large enough. In this case, since all connected components of Ω are bounded in the direction e, pulsating fronts in the directions $\pm e$ for problem (1.8) in Ω make no sense even if, under the notation of part b), the solutions u_n can be shifted to converge to a non-trivial solution u of (1.8) in $\mathbb{R} \times \overline{\Omega}$: what happens is that, in each connected component of Ω_i, u is just a time connection between 0 and a non-trivial steady state.

On the other hand, Theorem 1.4 also gives some simple geometrical conditions, of the types (1.20) or (1.21), for non-blocking in the directions $\pm e$. These conditions mean that Ω contains a slab which is not orthogonal to e, or contains a cylinder in a direction which is not orthogonal to e. We do not know however if these conditions are optimal, even when A is constant. Lastly, Theorem 1.4 shows the existence of pulsating fronts for problem (1.8) in Ω. Assume for instance that Ω is connected, that is $m = 1$ under notation (1.10). Then, there are pulsating traveling fronts, in the usual sense, in the direction e, connecting 0 to a non-trivial periodic stationary solution of (1.8). Furthermore, if F is of the KPP type (1.4) in Ω, the limiting state is unique and is then equal to the function $p = p_\infty$ given in Theorems 1.2 and 1.3 (see Remark 3.1 below and the end of the proof of Theorem 1.4). However, Theorem 1.4 holds for general monostable functions F which may not be of the KPP type and it gives the first result about the existence of pulsating fronts with zero Dirichlet boundary condition in periodic domains (which may not be cylinders).

Outline of the paper. Section 2 is devoted to the proof of Proposition 1.1 and Theorem 1.2 about the existence of minimal non-trivial stationary solutions p of problems (1.7) and (1.13) respectively, and about the large-time behavior of the solutions u of the Cauchy problems (1.5) and (1.14). Section 3 is concerned with the proof of Theorem 1.3 and the relationship between the minimal solutions p_n of problems (1.7) with nonlinearities f_n and the minimal solution p of problem (1.13). Lastly, in Section 4, we do the proof of Theorem 1.4 and make clear the role of the geometrical condition (1.19) in the blocking process as $n \to +\infty$.

2 Minimal stationary solutions and large-time behavior for the Cauchy problems (1.5) and (1.14)

In the first part of this section, we first deal with the elliptic and parabolic problems (1.7) and (1.5) set in the whole space $\mathbb{R}^N$ with the assumption (1.3) on the nonlinearity f. Namely, we do the proof of Proposition 1.1. It is based on the elliptic and parabolic maximum principles and on the construction of suitable subsolutions. Since some parts of the proof are quite similar to some arguments used in [5] and [7], they will only be sketched. In the second part of this section, we will be concerned with the stationary and Cauchy problems (1.13) and (1.14) posed in the set Ω with zero Dirichlet boundary condition. That is, we will do the proof of Theorem 1.2, which will itself be inspired by that of Proposition 1.1, but additional difficulties arise.

Proof of Proposition 1.1. Let φ be the unique periodic solution of (1.6) such that $\max_{\mathbb{R}^N}\varphi = 1$. Since the principal periodic eigenvalue λ_1 of (1.6) is assumed to be negative, one can fix $\varepsilon_0 \in (0, M]$ so that $f(x, s) \geq \zeta(x)\, s + (\lambda_1/2)s$ for all $(x, s) \in \mathbb{R}^N \times [0, \varepsilon_0]$. Now, for any $\varepsilon \in (0, \varepsilon_0]$, there holds

$$-\nabla\cdot(A(x)\nabla(\varepsilon\varphi)) - f(x, \varepsilon\varphi) \leq -\varepsilon\nabla\cdot(A(x)\nabla\varphi) - \zeta(x)\varepsilon\varphi - \frac{\lambda_1}{2}\varepsilon\varphi = \frac{\lambda_1}{2}\varepsilon\varphi < 0 \tag{2.1}$$

for all $x \in \mathbb{R}^N$. In other words, the functions $\varepsilon\,\varphi$ are strict subsolutions of (1.7) for all $\varepsilon \in (0, \varepsilon_0]$.

Let now U be the solution of the Cauchy problem (1.5) with initial datum $U_0 = \varepsilon_0\varphi$. Since $0 < U_0 \leq M$ and $f(\cdot, M) \leq 0$ in $\mathbb{R}^N$ and since U_0 is a subsolution of (1.7), it follows that

$$\varepsilon_0\varphi(x) \leq U(t, x) \leq M \text{ for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$$

and that U is nondecreasing with respect to t. Furthermore, by uniqueness for the Cauchy problem (1.5), $U(t, \cdot)$ is periodic in $\mathbb{R}^N$ for each $t \geq 0$. From standard parabolic estimates, it follows then that

$$U(t, x) \to p(x) \text{ as } t \to +\infty \text{ uniformly with respect to } x \in \mathbb{R}^N,$$

where p is a $C^{2,\alpha}(\mathbb{R}^N)$ periodic solution of (1.7) such that $0 < \varepsilon_0\varphi = U_0 \leq p \leq M$.

Let us then show that p is the minimal positive solution of (1.7) (in the class of all positive solutions of (1.7), which are not a priori assumed to be periodic). Let q be any positive solution of (1.7). Let $\lambda_{1,B(y,R),D}$ denote the principal eigenvalue of the operator

$$-\nabla\cdot(A(x)\nabla) - \zeta(x)$$

in the open Euclidean ball $B(y, R)$ of center $y \in \mathbb{R}^N$ and radius $R > 0$, with zero Dirichlet boundary condition on $\partial B(y, R)$. For each point $y \in \mathbb{R}^N$ and $R > 0$, the principal eigenvalue $\lambda_{1,B(y,R),D}$ is characterized by the existence of a function $\varphi_{y,R}$ of class $C^{2,\alpha}(\overline{B(y, R)})$, solving

$$\begin{cases} -\nabla\cdot(A(x)\nabla\varphi_{y,R}) - \zeta(x)\varphi_{y,R} = \lambda_{1,B(y,R),D}\,\varphi_{y,R} & \text{in } \overline{B(y, R)}, \\ \varphi_{y,R} > 0 & \text{in } B(y, R), \\ \varphi_{y,R} = 0 & \text{on } \partial B(y, R). \end{cases}$$

Up to normalization, one can assume that $\max_{\overline{B(y,R)}}\varphi_{y,R} = 1$, and the functions $\varphi_{y,R}$ are then unique. As done in [7], there holds

$$\lambda_{1,B(y,R),D} \to \lambda_1 \text{ as } R \to +\infty,$$

uniformly with respect to $y \in \mathbb{R}^N$. Since $\lambda_1 < 0$, one can then fix $R > 0$ large enough so that $\lambda_{1,B(y,R),D} < \lambda_1/2$ for all $y \in \mathbb{R}^N$. Thus, for each $y \in \mathbb{R}^N$ and $\varepsilon \in (0, \varepsilon_0]$, the function $\varepsilon\varphi_{y,R}$ satisfies

$$\begin{aligned} -\nabla\cdot(A(x)\nabla(\varepsilon\varphi_{y,R})) - f(x, \varepsilon\varphi_{y,R}) &\le -\varepsilon\nabla\cdot(A(x)\nabla\varphi_{y,R}) - \zeta(x)\varepsilon\varphi_{y,R} - \frac{\lambda_1}{2}\varepsilon\varphi_{y,R} \\ &= \Big(\lambda_{1,B(y,R),D} - \frac{\lambda_1}{2}\Big)\varepsilon\varphi_{y,R} \\ &< 0 \end{aligned} \tag{2.2}$$

in $B(y, R)$. In other words, the functions $\varepsilon\varphi_{y,R}$ are strict subsolutions of (1.7) in the balls $B(y, R)$ for all $\varepsilon \in (0, \varepsilon_0]$. Now, fix $y \in \mathbb{R}^N$ and observe that $\min_{\overline{B(y,R)}} q > 0$ by continuity of q. It follows then that

$$\varepsilon_y^* := \sup\Big\{\varepsilon \in (0, \varepsilon_0],\ \varepsilon\varphi_{y,R} \le q \text{ in } \overline{B(y, R)}\Big\}$$

is positive. We shall prove that $\varepsilon_y^* = \varepsilon_0$. Assume not. Then $0 < \varepsilon_y^* < \varepsilon_0$ and $\varepsilon_y^*\varphi_{y,R} \le q$ in $\overline{B(y, R)}$ with equality somewhere in $\overline{B(y, R)}$. Since $q > 0$ and $\varphi_{y,R} = 0$ on $\partial B(y, R)$, the functions $\varepsilon_y^*\varphi_{y,R}$ and q are equal somewhere at an interior point, in $B(y, R)$. But $\varepsilon_y^*\varphi_{y,R}$ is a subsolution of (1.7), from (2.2). Since f is (at least) Lipschitz-continuous locally with respect to the second variable, uniformly in x, it follows from the strong elliptic maximum principle that $\varepsilon_y^*\varphi_{y,R} = q$ in $B(y, R)$, which is impossible since the inequality (2.2) is strict. Therefore, $\varepsilon_y^* = \varepsilon_0$ for all $y \in \mathbb{R}^N$ and, in particular,

$$q(y) \ge \varepsilon_0\varphi_{y,R}(y) \text{ for all } y \in \mathbb{R}^N.$$

But, by uniqueness of the principal eigenfunctions $\varphi_{y,R}$ and by periodicity of A and ζ, the function $y \mapsto \varphi_{y,R}(y)$ is continuous and periodic in $\mathbb{R}^N$. Since it is positive, one gets that $\min_{y\in\mathbb{R}^N}\varphi_{y,R}(y) > 0$. Therefore, $\inf_{\mathbb{R}^N} q > 0$.

Define now

$$\varepsilon^* = \sup\Big\{\varepsilon \in (0, \varepsilon_0],\ \varepsilon\varphi \le q \text{ in } \mathbb{R}^N\Big\},$$

where we recall that φ is the unique periodic solution of (1.6) such that $\max_{\mathbb{R}^N}\varphi = 1$. Since q is bounded from below in the whole space $\mathbb{R}^N$ by a positive constant and since φ is bounded, one has $\varepsilon^* > 0$. Assume that $\varepsilon^* < \varepsilon_0$. Then $\varepsilon^*\varphi \le q$ in $\mathbb{R}^N$ and there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\varepsilon^*\varphi(x_k) - q(x_k) \to 0 \text{ as } k \to +\infty.$$

By writing $x_k = x_k' + x_k''$ with $x_k' \in L_1\mathbb{Z} \times \cdots L_N\mathbb{Z}$ and $x_k'' \in [0, L_1] \times \cdots \times [0, L_N]$, it follows that the functions $q_k(x) = q(x + x_k')$ converge, up to extraction of a subsequence, to a solution q_∞ of (1.7) such that $\varepsilon^*\varphi \le q_\infty$ in $\mathbb{R}^N$ with equality somewhere in $\mathbb{R}^N$. As above, one concludes that $\varepsilon^*\varphi = q_\infty$ in $\mathbb{R}^N$, which is impossible since $\varepsilon^*\varphi$ is a strict subsolution of (1.7), from (2.1). Therefore, $\varepsilon^* = \varepsilon_0$, whence $\varepsilon_0\varphi \le q$ in $\mathbb{R}^N$. The parabolic maximum principle implies that

$$U(t,x) \le q(x) \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

where we recall that U denotes the solution of (1.5) with initial datum $\varepsilon_0\varphi$. By passing to the limit as $t \to +\infty$, one gets that

$$p(x) \le q(x) \text{ for all } x \in \mathbb{R}^N.$$

Finally, let $u_0 : \mathbb{R}^N \to [0, M]$ be a uniformly continuous function which is not identically equal to 0, and let u denote the solution of (1.5) with initial datum u_0. The maximum principle implies that $0 \le u(t,x) \le M$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$, and $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$. With the same notation as above, there exists then $\varepsilon \in (0, \varepsilon_0]$ such that

$$\varepsilon\,\varphi_{0,R} \le u(1,\cdot) \text{ in } \overline{B(0,R)},$$

where we recall that $R > 0$ was chosen so that $\lambda_{1,B(y,R),D} < \lambda_1/2$ for all $y \in \mathbb{R}^N$. Let v be the solution of (1.5) with initial datum

$$v_0(x) = \begin{cases} \varepsilon\,\varphi_{0,R}(x) & \text{if } x \in \overline{B(0,R)}, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash \overline{B(0,R)}. \end{cases}$$

Since $0 \le v_0 \le u(1,\cdot) \le M$ in $\mathbb{R}^N$, there holds

$$0 \le v(t,x) \le u(t+1,x) \le M \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

Furthermore, since v_0 is a subsolution of (1.7) because of (2.2) and $v_0 = 0$ in $\mathbb{R}^N \backslash \overline{B(0,R)}$, it follows from the maximum principle that v is nondecreasing with respect to t. Hence, from standard parabolic estimates, one gets that

$$v(t,x) \to v_\infty(x) \text{ as } t \to +\infty \text{ locally uniformly in } x \in \mathbb{R}^N,$$

where v_∞ is a solution of (1.7) satisfying $v_0 \le v_\infty \le M$ in $\mathbb{R}^N$. Notice in particular that v_∞ is positive in $\mathbb{R}^N$ from the strong maximum principle, since v_0 is nonnegative and not identically equal to 0. But the previous paragraphs yield then $v_\infty \ge p$. Therefore,

$$\liminf_{t \to +\infty} u(t,x) \ge p(x)$$

locally uniformly in $x \in \mathbb{R}^N$. Lastly, if $u_0 \le p$, then $u(t,x) \le p(x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$, whence $u(t,x) \to p(x)$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$. The proof of Proposition 1.1 is thereby complete. □

Let us now turn to the proof of Theorem 1.2. Some of the ideas of the proof of Proposition 1.1 can be adapted. However, the case of problems (1.13) and (1.14) in Ω is substantially more involved than the case of the whole space $\mathbb{R}^N$, mainly due to the fact that zero Dirichlet boundary condition is imposed on $\partial\Omega$ and the connected components of Ω may be bounded or unbounded.

Proof of Theorem 1.2. Remember that the sets Ω_i given in (1.10) are all periodic and pairwise disjoint. We first work in each set Ω_i for which $\lambda_{1,\Omega_i,D} < 0$, that is $i \in I_-$. We claim that, for each such index $i \in I_-$, there exists a periodic solution $\widetilde{p}_i \in C^{2,\alpha}(\overline{\Omega_i})$ of the stationary problem

$$\begin{cases} -\nabla\cdot(A(x)\nabla\widetilde{p}_i) &=& F(x,\widetilde{p}_i(x)) \;\; \text{in } \overline{\Omega_i}, \\ \widetilde{p}_i &=& 0 \qquad\qquad\quad\; \text{on } \partial\Omega_i, \\ \widetilde{p}_i &>& 0 \qquad\qquad\quad\; \text{in } \Omega_i. \end{cases} \tag{2.3}$$

Indeed, let $\widetilde{\varphi}_i$ be the principal periodic eigenfunction of the operator $-\nabla\cdot(A(x)\nabla)-\zeta(x)$ in Ω_i with zero Dirichlet boundary condition on $\partial\Omega_i$. That is, the function $\widetilde{\varphi}_i$ is periodic, of class $C^{2,\alpha}(\overline{\Omega_i})$, and it solves

$$\begin{cases} -\nabla\cdot(A(x)\nabla\widetilde{\varphi}_i)-\zeta(x)\widetilde{\varphi}_i &=& \lambda_{1,\Omega_i,D}\,\widetilde{\varphi}_i \;\; \text{in } \overline{\Omega_i}, \\ \widetilde{\varphi}_i &=& 0 \qquad\qquad\;\; \text{on } \partial\Omega_i, \\ \widetilde{\varphi}_i &>& 0 \qquad\qquad\;\; \text{in } \Omega_i. \end{cases} \tag{2.4}$$

Up to normalization, one can assume that $\max_{\overline{\Omega_i}}\widetilde{\varphi}_i = 1$. Now, as in the proof of Proposition 1.1, since $\lambda_{1,\Omega_i,D} < 0$, there exists $\varepsilon_0 \in (0,M]$ such that, for any $\varepsilon \in (0,\varepsilon_0]$, the function $\varepsilon\widetilde{\varphi}_i$ is a strict subsolution of (2.3), namely

$$-\nabla\cdot(A(x)\nabla(\varepsilon\widetilde{\varphi}_i)) - F(x,\varepsilon\widetilde{\varphi}_i(x)) < 0 \text{ in } \Omega_i, \tag{2.5}$$

together with $\varepsilon\widetilde{\varphi}_i = 0$ on $\partial\Omega_i$ and $\varepsilon\widetilde{\varphi}_i > 0$ in Ω_i. But since the constant M is a supersolution of this problem, the solution u_i of the Cauchy problem

$$\begin{cases} (u_i)_t - \nabla\cdot(A(x)\nabla u_i) &=& F(x,u_i), \;\; t>0,\; x\in\overline{\Omega_i}, \\ u_i(t,x) &=& 0, \qquad\qquad t>0,\; x\in\partial\Omega_i, \\ u_i(0,x) &=& \varepsilon_0\widetilde{\varphi}_i(x), \;\;\; x\in\Omega_i, \end{cases} \tag{2.6}$$

is such that $\varepsilon_0\widetilde{\varphi}_i(x) \le u_i(t,x) \le M$ for all $(t,x)\in(0,+\infty)\times\overline{\Omega_i}$ and u_i is nondecreasing in t and periodic in x in $\overline{\Omega_i}$. Therefore, there exists a periodic $C^{2,\alpha}(\overline{\Omega_i})$ solution $\widetilde{p}_i$ of (2.3) such that $u_i(t,x)\to\widetilde{p}_i(x)$ as $t\to+\infty$, uniformly in $x\in\overline{\Omega_i}$.

Let now $\widetilde{q}_i$ be any classical bounded solution of (2.3) and let us prove that $\widetilde{q}_i \ge \widetilde{p}_i$ in $\overline{\Omega_i}$. By definition of Ω_i, the set ω_i is one of its connected components, and any of its connected components is of the type $\omega_i + k$ for some $k \in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$. Two cases may then occur: either ω_i is bounded, or ω_i is unbounded.

Case 1. Consider first the case when ω_i is bounded. Since $\widetilde{q}_i > 0$ in $\omega_i\,(\subset\Omega_i)$, $\widetilde{q}_i = 0$ on $\partial\omega_i\,(\subset\partial\Omega_i)$ and $F(\cdot,0)\equiv 0$, it follows from Hopf lemma and the compactness of $\partial\omega_i$ that

$$\max_{x\in\partial\omega_i}\frac{\partial\widetilde{q}_i}{\partial\nu}(x) < 0,$$

where ν denotes the outward unit normal on $\partial\Omega$. On the other hand, the principal eigenfunction $\widetilde{\varphi}_i$ of (2.4) is (at least) of class $C^1(\overline{\omega_i})$ and $\widetilde{\varphi}_i = 0$ on $\partial\omega_i$. Hence, the quantity

$$\varepsilon^* := \sup\big\{\varepsilon \in (0, \varepsilon_0],\ \varepsilon\, \widetilde{\varphi}_i \le \widetilde{q}_i \text{ in } \overline{\omega_i}\big\}$$

is a positive real number, belonging to the interval $(0, \varepsilon_0]$. Furthermore, $\varepsilon^*\widetilde{\varphi}_i \le \widetilde{q}_i$ in $\overline{\omega_i}$. Since $\varepsilon^*\widetilde{\varphi}_i$ is a strict subsolution in $\omega_i \subset \Omega_i$, in the sense of (2.5), the strong maximum principle and the Hopf lemma imply that $\varepsilon^*\widetilde{\varphi}_i < \widetilde{q}_i$ in ω_i and

$$\frac{\partial \widetilde{q}_i}{\partial \nu} < \varepsilon^* \frac{\partial \widetilde{\varphi}_i}{\partial \nu} \quad \text{on } \partial\omega_i.$$

Therefore, there exists $\eta_0 > 0$ such that $(\varepsilon^* + \eta)\, \widetilde{\varphi}_i \le \widetilde{q}_i$ in $\overline{\omega_i}$ for all $\eta \in [0, \eta_0]$. The definition of ε^* then yields $\varepsilon^* = \varepsilon_0$, whence $\varepsilon_0\widetilde{\varphi}_i \le \widetilde{q}_i$ in $\overline{\omega_i}$. The same argument can be repeated in $\omega_i + k$ for all $k \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$. Therefore, $\varepsilon_0\widetilde{\varphi}_i \le \widetilde{q}_i$ in $\overline{\Omega_i}$. By comparing $\widetilde{q}_i$ with the solution u_i of the Cauchy problem (2.6), it follows then as in the proof of Proposition 1.1 that

$$\widetilde{p}_i \le \widetilde{q}_i \quad \text{in } \overline{\Omega_i}. \tag{2.7}$$

Case 2. Consider now the case when ω_i is unbounded. For all $y \in \omega_i$ and $R > 0$, define

$$\omega_{i,y,R} = \big\{z \in \omega_i,\ d_\Omega(y, z) < R\big\},$$

where d_Ω denotes the geodesic distance inside Ω, and set

$$\lambda_{1,\omega_{i,y,R},D} = \min_{\phi \in H^1_0(\omega_{i,y,R}) \setminus \{0\}} \frac{\displaystyle\int_{\omega_{i,y,R}} A\nabla\phi \cdot \nabla\phi - \zeta\phi^2}{\displaystyle\int_{\omega_{i,y,R}} \phi^2}. \tag{2.8}$$

Actually, $\lambda_{1,\omega_{i,y,R},D}$ is the smallest eigenvalue of the operator $-\nabla \cdot (A\nabla) - \zeta$ in $\omega_{i,y,R}$ with zero Dirichlet boundary condition (that is, in the $H^1_0(\omega_{i,y,R})$ sense), but, since $\partial\omega_{i,y,R}$ may not be smooth in general, the eigenvalue $\lambda_{1,\omega_{i,y,R},D}$ may not be associated with $C^1(\overline{\omega_{i,y,R}})$ eigenfunctions. We first claim that

$$\limsup_{R \to +\infty} \Big(\sup_{y \in \omega_i} \lambda_{1,\omega_{i,y,R},D}\Big) < 0.$$

To do so, let $\widetilde{\rho} : \mathbb{R} \to [0, 1]$ be a $C^\infty(\mathbb{R})$ function such that $\widetilde{\rho} = 1$ on $(-\infty, -1]$ and $\widetilde{\rho} = 0$ on $[0, +\infty)$ and, for all $y \in \omega_i$ and $R > 0$, denote

$$\rho_{y,R}(x) = \widetilde{\rho}\big(d_\Omega(x, y) - R\big) \quad \text{for all } x \in \omega_i.$$

These functions $\rho_{y,R}$ are then is $W^{1,\infty}(\omega_i)$. For every $y \in \omega_i$ and $R > 1$, the restriction of the function $\widetilde{\varphi}_i\rho_{y,R}$ to $\omega_{i,y,R}$ belongs to $H^1_0(\omega_{i,y,R})\backslash\{0\}$, whence

$$\begin{aligned}\lambda_{1,\omega_{i,y,R},D} &\le \frac{\displaystyle\int_{\omega_{i,y,R}} A\nabla(\widetilde{\varphi}_i\rho_{y,R})\cdot\nabla(\widetilde{\varphi}_i\rho_{y,R})-\zeta(\widetilde{\varphi}_i\rho_{y,R})^2}{\displaystyle\int_{\omega_{i,y,R}}(\widetilde{\varphi}_i\rho_{y,R})^2}\\ &\le \frac{\displaystyle\int_{\omega_{i,y,R}} \rho_{y,R}A\nabla\widetilde{\varphi}_i\cdot\nabla(\widetilde{\varphi}_i\rho_{y,R})-\zeta(\widetilde{\varphi}_i\rho_{y,R})^2}{\displaystyle\int_{\omega_{i,y,R}}(\widetilde{\varphi}_i\rho_{y,R})^2}+\frac{M\,|\omega_{i,y,R}\backslash\omega_{i,y,R-1}|}{\displaystyle\int_{\omega_{i,y,R}}(\widetilde{\varphi}_i\rho_{y,R})^2},\end{aligned}$$

where

$$M=(1+\|\nabla\widetilde{\varphi}_i\|_{L^\infty(\Omega_i)})\times\max_{x\in\overline{\Omega},\,|\xi|=1,\,|\xi'|=1}(A(x)\xi\cdot\xi')$$

is a positive constant which does not depend on y or R, and $|\omega_{i,y,R}\backslash\omega_{i,y,R-1}|$ denotes the Lebesgue measure of $\omega_{i,y,R}\backslash\omega_{i,y,R-1}$. By integrating by parts, it follows then from (2.4) that

$$\lambda_{1,\omega_{i,y,R},D}\le\lambda_{1,\Omega_i,D}+\frac{2\,M\,|\omega_{i,y,R}\backslash\omega_{i,y,R-1}|}{\displaystyle\int_{\omega_{i,y,R}}(\widetilde{\varphi}_i\rho_{y,R})^2}.$$

Since $\widetilde{\varphi}_i$ is periodic and positive in Ω_i (and then uniformly away from 0 in each non-empty set of the type

$$\omega_i^\delta:=\big\{x\in\omega_i,\ d(x,\partial\omega_i)>\delta\big\}$$

with $\delta>0$) and since Ω (and hence ω_i) has a smooth boundary, it follows that

$$\liminf_{R\to+\infty}\Big(\inf_{y\in\omega_i}|\omega_{i,y,R-1}|^{-1}\int_{\omega_{i,y,R}}(\widetilde{\varphi}_i\rho_{y,R})^2\Big)>0,$$

while $\limsup_{R\to+\infty}\big(\sup_{y\in\omega_i}|\omega_{i,y,R-1}|^{-1}|\omega_{i,y,R}\backslash\omega_{i,y,R-1}|\big)=0$. Remember that $\lambda_{1,\Omega_i,D}<0$. Therefore, there exists $R_0>1$ such that

$$\forall\,R\ge R_0,\ \forall\,y\in\omega_i,\quad \lambda_{1,\omega_{i,y,R},D}<\frac{\lambda_{1,\Omega_i,D}}{2}. \tag{2.9}$$

Let now $\delta>0$ be any positive constant such that $\omega_i^\delta\neq\emptyset$ and let us show that $\inf_{\omega_i^\delta}\widetilde{q}_i>0$. Assume not and let $\varepsilon_i>0$ be such that $F(x,s)\ge\zeta(x)s+(\lambda_{1,\Omega_i,D}/2)s$ for all $(x,s)\in\overline{\Omega}\times[0,\varepsilon_i]$. There is then a sequence $(x_n)_{n\in N}$ in ω_i^δ such that $\widetilde{q}_i(x_n)\to0$ as $n\to+\infty$. Since $\widetilde{q}_i\ge0$ in $\overline{\omega_i}$ and $\widetilde{q}_i=0$ on $\partial\omega_i$, it follows from Harnack inequality that

$$\max_{\overline{\omega_{i,x_{n_0},R_0}}}\widetilde{q}_i\le\varepsilon_i\ \text{ for some } n_0\in\mathbb{N} \text{ large enough.}$$

In particular,

$$-\nabla\cdot(A(x)\nabla\widetilde{q}_i)-\zeta(x)\widetilde{q}_i\ge\frac{\lambda_{1,\Omega_i,D}}{2}\,\widetilde{q}_i\ \text{ in } \omega_{i,x_{n_0},R_0}. \tag{2.10}$$

On the other hand, from (2.8) and (2.9), and owing to the definition of $H_0^1(\omega_{i,x_{n_0},R_0})$, there is $\phi\in C_c^1(\omega_{i,x_{n_0},R_0})\backslash\{0\}$ (with a compact support which is included in ω_{i,x_{n_0},R_0}) such that

$$R[\phi]:=\frac{\displaystyle\int_{\omega_{i,x_{n_0},R_0}}A\nabla\phi\cdot\nabla\phi-\zeta\phi^2}{\displaystyle\int_{\omega_{i,x_{n_0},R_0}}\phi^2}<\frac{\lambda_{1,\Omega_i,D}}{2}.$$

Now, let ω' be any bounded open set of class $C^{2,\alpha}$, containing the support of ϕ, and such that $\overline{\omega'}\subset\omega_{i,x_{n_0},R_0}$. It follows that $\lambda_{1,\omega',D}\le R[\phi]<\lambda_{1,\Omega_i,D}/2$. There is then a nonnegative and nontrivial function $\varphi'\in C^{2,\alpha}(\overline{\omega'})$ solving

$$-\nabla\cdot(A(x)\nabla\varphi')-\zeta(x)\varphi'=\lambda_{1,\omega',D}\varphi'\le\frac{\lambda_{1,\Omega_i,D}}{2}\varphi'\ \text{ in }\omega' \tag{2.11}$$

with $\varphi'=0$ on $\partial\omega'$. Notice that φ' may not be positive in ω' since ω' may not be connected. But φ' is positive at least in one connected component ω'' of ω'. Since $\min_{\overline{\omega''}}\widetilde{q}_i>0$ and $\varphi'=0$ on $\partial\omega''$, it follows from (2.10), (2.11) and the strong maximum principle that $\varepsilon\varphi'\le\widetilde{q}_i$ in $\overline{\omega''}$ for all $\varepsilon>0$, which is clearly impossible. One has then reached a contradiction. Hence there holds

$$\inf_{\omega_i^\delta}\widetilde{q}_i>0\ \text{ for all }\delta>0\text{ such that }\omega_i^\delta\neq\emptyset. \tag{2.12}$$

It follows then from (2.12), together with the Hopf lemma and the global smoothness of $\partial\omega_i$, that $\sup_{\partial\omega_i}\frac{\partial\widetilde{q}_i}{\partial\nu}<0$. Therefore, the quantity

$$\varepsilon^*:=\sup\big\{\varepsilon\in(0,\varepsilon_0],\ \varepsilon\widetilde{\varphi}_i\le\widetilde{q}_i\text{ in }\overline{\omega_i}\big\}$$

is a positive real number. From (2.5) and the strong maximum principle, there holds $\varepsilon^*\widetilde{\varphi}_i<\widetilde{q}_i$ in ω_i. Furthermore, we claim that

$$\inf_{\omega_i^\delta}(\widetilde{q}_i-\varepsilon^*\widetilde{\varphi}_i)>0\ \text{ for all }\delta>0\text{ such that }\omega_i^\delta\neq\emptyset. \tag{2.13}$$

Assume not. Then there exist $\delta>0$ such that $\omega_i^\delta\neq\emptyset$ and a sequence $(y_n)_{n\in\mathbb{N}}$ in ω_i^δ such that $\widetilde{q}_i(y_n)-\varepsilon^*\widetilde{\varphi}_i(y_n)\to0$ as $n\to+\infty$. Write $y_n=y_n'+y_n''$ where $y_n'\in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$ and $y_n''\in C$. Notice in particular that $d(y_n'',\partial\Omega)=d(y_n'',\partial\Omega_i)>\delta$. Up to extraction of a subsequence, one can assume that $y_n''\to y_\infty\in\Omega_i$ as $n\to+\infty$ with

$$d(y_\infty,\partial\Omega)=d(y_\infty,\partial\Omega_i)\ge\delta,$$

and that the functions $x\mapsto\widetilde{q}_i(x+y_n')$ defined in $\overline{\Omega_i}$ converge in $C^2_{loc}(\overline{\Omega_i})$ to a solution $\overline{q}_i$ of

$$-\nabla\cdot(A(x)\nabla\overline{q}_i)=F(x,\overline{q}_i(x))\text{ in }\overline{\Omega_i}$$

such that $\overline{q}_i\ge\varepsilon^*\widetilde{\varphi}_i$ in $B(y_\infty,\delta)\subset\Omega_i$ with equality at y_∞. The strong maximum principle and (2.5) lead to a contradiction. Thus, the claim (2.13) holds. As above, it follows then from Hopf lemma and the global smoothness of $\partial\omega_i$ that $\sup_{\partial\omega_i}\frac{\partial(\widetilde{q}_i-\varepsilon^*\widetilde{\varphi}_i)}{\partial\nu}<0$ and that there exists

$\eta_0 > 0$ such that $(\varepsilon^* + \eta)\, \widetilde{\varphi}_i \le \widetilde{q}_i$ in $\overline{\omega_i}$ for all $\eta \in [0, \eta_0]$. Therefore, $\varepsilon^* = \varepsilon_0$, whence $\varepsilon_0 \widetilde{\varphi}_i \le \widetilde{q}_i$ in $\overline{\omega_i}$ and then in $\overline{\Omega_i}$ by repeating the argument in $\omega_i + k$ for all k in $L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$. Finally, by comparing $\widetilde{q}_i$ with the solution u_i of the Cauchy problem (2.6), the conclusion (2.7) follows.

Conclusion of the proof. Define the function p in $\overline{\Omega}$ by

$$p = \begin{cases} \widetilde{p}_i & \text{in all the sets } \overline{\Omega_i} \text{ with } i \in I_-, \\ 0 & \text{in all the sets } \overline{\Omega_i} \text{ with } i \notin I_-. \end{cases}$$

The function p is periodic, of class $C^{2,\alpha}(\overline{\Omega})$, and it solves (1.13). Furthermore, it follows from the previous steps that any bounded solution q of (1.13) is such that $q \ge p$ in $\overline{\Omega}$. Lastly, let $u_0 : \overline{\Omega} \to [0, M]$ be any uniformly continuous function such that $u_0 \not\equiv 0$ in $\overline{\Omega}$, let u be the solution of the Cauchy problem (1.14) and let ω be a connected component of Ω intersecting the support of u_0. We shall prove that

$$\liminf_{t \to +\infty} \Big(\min_{x \in K} (u(t,x) - p(x)) \Big) \ge 0 \tag{2.14}$$

for any compact set $K \subset \overline{\omega}$. Since $0 \le u(t,\cdot)\,(\le M)$ in $\overline{\Omega}$ for all $t > 0$ and $p = 0$ in $\overline{\Omega_i}$ for all $i \notin I_i$, it is sufficient to consider the case when $\omega = \omega_i + k$ for some $i \in I_-$ and some $k \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$.

If ω is bounded, then $u(1,\cdot) > 0$ in ω and $\max_{\partial\omega} \frac{\partial u(1,\cdot)}{\partial \nu} < 0$ from the strong parabolic maximum principle. Therefore, $u(1,\cdot) \ge \varepsilon \widetilde{\varphi}_i$ in $\overline{\omega}$ for some $\varepsilon \in (0, \varepsilon_0]$ and

$$u(t+1, x) \ge v(t,x) \text{ for all } (t,x) \in (0,+\infty) \times \overline{\omega},$$

where v is the solution of the Cauchy problem (2.6) in $\overline{\omega}$ with initial datum $\varepsilon \widetilde{\varphi}_i$ in $\overline{\omega}$ and zero Dirichlet boundary condition on $\partial\omega$. Owing to (2.5), $v(t,x)$ is increasing with respect to t (and bounded from above by the constant M), and it converges as $t \to +\infty$ uniformly in $\overline{\omega}$ to a solution w of (2.3) in $\overline{\omega}$ such that $w \ge \varepsilon \widetilde{\varphi}_i$ in $\overline{\omega}$ (whence $w > 0$ in ω) and $w = 0$ on $\partial\omega$. It follows as in the study of case 1 above that $w \ge p$ in $\overline{\omega}$, which yields (2.14).

Consider now the case when ω is unbounded. Without loss of generality, up to a translation of the origin, one can assume that $k = 0$ and $\omega = \omega_i$. Choose any point y_0 in ω and, from (2.9), let $R_0 > 0$ be such that $\lambda_{1,\omega_{i,y_0,R_0},D} < 0$. As above, there is then a function $\phi \in C^1_c(\omega_{i,y_0,R_0}) \backslash \{0\}$ such that

$$R'[\phi] := \frac{\displaystyle\int_{\omega_{i,y_0,R_0}} A\nabla\phi \cdot \nabla\phi - \zeta\phi^2}{\displaystyle\int_{\omega_{i,y_0,R_0}} \phi^2} < 0$$

and, if ω' is any bounded open set of class $C^{2,\alpha}$ containing the support of ϕ and such that $\overline{\omega'} \subset \omega_{i,y_0,R_0} \subset \omega$, there holds $\lambda_{1,\omega',D} \le R'[\phi] < 0$. There is then a nonnegative and nontrivial function $\varphi' \in C^{2,\alpha}(\overline{\omega'})$ such that

$$-\nabla \cdot (A(x)\nabla\varphi') - \zeta(x)\varphi' = \lambda_{1,\omega',D}\varphi' \text{ in } \omega'$$

with $\varphi' = 0$ on $\partial\omega'$. Therefore, the function $\varepsilon'\varphi'$ is a subsolution of (2.3) in ω' for $\varepsilon' > 0$ small enough and one can also assume without loss of generality that $\varepsilon'\varphi' \le u(1,\cdot)$ in the compact set $\overline{\omega'} \subset \omega$. Thus, there holds $u(t+1,x) \ge v(t,x)$ for all $(t,x) \in (0,+\infty)\times\overline{\omega}$, where v is the solution of the Cauchy problem (2.6) in $\overline{\omega}$ with initial datum $v_0 = \varepsilon'\varphi'$ in $\overline{\omega'}$ and $v_0 = 0$ in $\overline{\omega}\backslash\overline{\omega'}$, and zero Dirichlet boundary condition on $\partial\omega$. But $v(t,x)$ is increasing with respect to t and bounded from above by M. It converges locally uniformly in $\overline{\omega}$ to a solution w of (2.3) in $\overline{\omega}$, such that $w \ge v_0$ in $\overline{\omega}$ (whence $w > 0$ in ω from the strong maximum principle). One concludes as in case 2 above that $w \ge p$ in $\overline{\omega}$, which leads to (2.14).

Lastly, observe that, if $u_0 \le p$ in $\overline{\Omega}$, then $u(t,\cdot) \le p$ in $\overline{\Omega}$ for all $t > 0$. Hence, (2.14) implies that $u(t,x) \to p(x)$ as $t \to +\infty$ uniformly in any compact subset $K \subset \overline{\omega}$, where ω is any connected component of Ω intersecting the support of u_0. The proof of Theorem 1.2 is thereby complete. □

3 Relationship between the problems (1.7) with nonlinearities f_n and the problem (1.18)

This section is devoted to the proof of Theorem 1.3. By using variational arguments, H^1 a priori estimates and Rellich's theorem, we prove the monotonicity and the convergence of the principal periodic eigenvalues of the linearized operators in $\mathbb{R}^N$ associated with the functions f_n, to that of problem (1.11) with zero Dirichlet boundary condition on $\partial\Omega$. Then, we show the monotonicity and the convergence of the functions p_n to a solution $p_\infty \ge p$ of (1.18). The minimality of each solution p_n and of p will also be used.

Proof of Theorem 1.3. Let, for each $n \in \mathbb{N}$, $\lambda_{1,n}$ and φ_n be the principal eigenvalue and periodic eigenfunction solving (1.17). Let $\lambda_{1,D}$ and φ solve (1.11), where one can always assume that $\varphi > 0$ in each Ω_i with $\lambda_{1,\Omega_i,D} = \lambda_{1,D}$, that is $i \in I_{min}$. Call

$$H^1_{per}(\mathbb{R}^N) = \big\{\phi \in H^1_{loc}(\mathbb{R}^N),\ \phi \text{ is periodic}\big\},\quad L^2_{per}(\mathbb{R}^N) = \big\{\phi \in L^2_{loc}(\mathbb{R}^N),\ \phi \text{ is periodic}\big\}$$

and $C_0 = [0,L_1]\times\cdots\times[0,L_N]$. For each $n \in \mathbb{N}$, there holds

$$\lambda_{1,n} = \min_{\phi\in H^1_{per}(\mathbb{R}^N)\backslash\{0\}} R_n[\phi] = R_n[\varphi_n],$$

where

$$R_n[\phi] = \frac{\displaystyle\int_{C_0} A\nabla\phi\cdot\nabla\phi - \zeta_n\phi^2}{\displaystyle\int_{C_0}\phi^2}.$$

Since the sequence $(\zeta_n(x))_{n\in\mathbb{N}}$ is nonincreasing for each $x \in \mathbb{R}^N$, it follows that the sequence $(\lambda_{1,n})_{n\in\mathbb{N}}$ is nondecreasing.

We now claim that $\lambda_{1,n} < \lambda_{1,D}$ for each $n \in \mathbb{N}$. The proof is based on some standard comparison arguments, used in [7, 9]. We just sketch it here for the sake of completeness.

Assume that $\lambda_{1,n} \geq \lambda_{1,D}$ for some $n \in \mathbb{N}$. Pick any index $i \in I_{min}$. Since $\zeta = \zeta_n$ in $\overline{\Omega_i} \subset \overline{\Omega}$, there holds

$$-\nabla \cdot (A(x)\nabla\varphi_n) - \zeta(x)\varphi_n = \lambda_{1,n}\varphi_n \geq \lambda_{1,D}\varphi_n \text{ in } \overline{\Omega_i}$$

and $\min_{\overline{\Omega_i}} \varphi_n > 0$. In other words, the periodic function φ_n is a supersolution of the linear equation satisfied by the periodic function φ in Ω_i. Since $\varphi = 0$ on $\partial\Omega_i$ and φ is (at least) of class $C^2(\overline{\Omega_i})$, it follows from the strong elliptic maximum principle that the quantity

$$\varepsilon^* = \sup\left\{\varepsilon \in (0, +\infty),\ \varepsilon\varphi \leq \varphi_n \text{ in } \overline{\Omega_i}\right\}$$

is actually equal to $+\infty$. This is a contradiction since φ is positive in Ω_i. Therefore, $\lambda_{1,n} < \lambda_{1,D}$ for all $n \in \mathbb{N}$.

As a consequence, the sequence $(\lambda_{1,n})_{n\in\mathbb{N}}$ converges monotonically to a real number $\lambda_{1,\infty}$ such that $\lambda_{1,\infty} \leq \lambda_{1,D}$. Let us now show that $\lambda_{1,\infty} = \lambda_{1,D}$. Normalize here the eigenfunctions φ_n so that $\|\varphi_n\|_{L^2(C_0)} = 1$. It follows that

$$\int_{C_0} A\nabla\varphi_n \cdot \nabla\varphi_n = \lambda_{1,n} + \int_{C_0} \zeta_n\varphi_n^2 \leq \lambda_{1,\infty} + \int_{C_0} \zeta_0\varphi_n^2 \leq \lambda_{1,\infty} + \max_{\mathbb{R}^N} \zeta_0.$$

Thus, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $H^1(C_0)$. There exists then a function φ_∞ in $H^1_{per}(\mathbb{R}^N)$ such that, up to extraction of a subsequence, $\varphi_n \to \varphi_\infty$ weakly in $H^1_{per}(\mathbb{R}^N)$ and strongly in $L^2_{per}(\mathbb{R}^N)$. In particular, $\varphi_\infty \geq 0$ a.e. in $\mathbb{R}^N$ and $\|\varphi_\infty\|_{L^2(C_0)} = 1$. Let K be any compact set such that $K \subset (\mathbb{R}^N \backslash \overline{\Omega}) \cap C_0$. For all $n \in \mathbb{N}$, one has

$$\begin{aligned} -\Big(\max_K \zeta_n\Big)\int_K \varphi_n^2 \leq -\int_K \zeta_n\varphi_n^2 &= \lambda_{1,n} - \int_{C_0} A\nabla\varphi_n \cdot \nabla\varphi_n + \int_{C_0\backslash K} \zeta_n\varphi_n^2 \\ &\leq \lambda_{1,\infty} + \int_{C_0\backslash K} \zeta_0\varphi_n^2 \leq \lambda_{1,\infty} + \sup_{C_0\backslash K} |\zeta_0|, \end{aligned}$$

whence $\|\varphi_n\|_{L^2(K)} \to 0$ as $n \to +\infty$ from (1.16). Thus, $\varphi_\infty = 0$ a.e. in K, and then a.e. in $\mathbb{R}^N \backslash \Omega$ and $\|\varphi_\infty\|_{L^2(\Omega\cap C_0)} = 1$. Furthermore, since $\varphi_\infty \in H^1_{per}(\mathbb{R}^N)$, one gets that the restriction of φ_∞ to Ω belongs to $H^1_{0,per}(\Omega)$, that is the space of periodic $H^1_{loc}(\overline{\Omega})$ functions whose trace is equal to 0 on $\partial\Omega$. Lastly, observe that

$$\int_{\Omega\cap C_0} A\nabla\varphi_n \cdot \nabla\varphi_n \leq \int_{C_0} A\nabla\varphi_n \cdot \nabla\varphi_n = \lambda_{1,n} + \int_{C_0} \zeta_n\varphi_n^2 \leq \lambda_{1,\infty} + \int_{C_0} \zeta_0\varphi_n^2,$$

while

$$\int_{C_0} \zeta_0\varphi_n^2 \to \int_{C_0} \zeta_0\varphi_\infty^2 = \int_{\Omega\cap C_0} \zeta\varphi_\infty^2$$

as $n \to +\infty$. Therefore,

$$\int_{\Omega\cap C_0} A\nabla\varphi_\infty \cdot \nabla\varphi_\infty \leq \liminf_{n\to+\infty} \int_{\Omega\cap C_0} A\nabla\varphi_n \cdot \nabla\varphi_n \leq \lambda_{1,\infty} + \int_{\Omega\cap C_0} \zeta\varphi_\infty^2,$$

that is $R_\infty[\varphi_\infty] \le \lambda_{1,\infty} \le \lambda_{1,D}$, where the functional R_∞ is defined by

$$R_\infty[\phi] = \frac{\displaystyle\int_{\Omega\cap C_0} A\nabla\phi\cdot\nabla\phi - \zeta\phi^2}{\displaystyle\int_{\Omega\cap C_0} \phi^2}$$

for all $\phi \in H^1_{0,per}(\Omega)\backslash\{0\}$. But since $\min_{\phi\in H^1_{0,per}(\Omega)\backslash\{0\}} R_\infty[\phi] = \lambda_{1,D}$, one concludes that $\lambda_{1,\infty} = \lambda_{1,D}$. In other words, $\lambda_{1,n} \to \lambda_{1,D}$ as $n \to +\infty$.

In the sequel, assume now that $\lambda_{1,D} < 0$. Consequently, for each $n \in \mathbb{N}$, one has $\lambda_{1,n} < \lambda_{1,D} < 0$ and, from Proposition 1.1, there exists a minimal periodic solution p_n of

$$\left\{\begin{array}{rcll} -\nabla\cdot(A(x)\nabla p_n) & = & f_n(x,p_n) & \text{in } \mathbb{R}^N,\\ 0 & < & p_n \le M & \text{in } \mathbb{R}^N.\end{array}\right.$$

Fix any two integers $n \le m$. Since

$$-\nabla\cdot(A(x)\nabla p_n) - f_m(x,p_n) = f_n(x,p_n) - f_m(x,p_n) \ge 0 \text{ in } \mathbb{R}^N,$$

the function p_n is a supersolution for the equation satisfied by p_m. From the proof of Proposition 1.1, there exists $\varepsilon_m > 0$ such that all functions $\varepsilon\varphi_m$ with $\varepsilon \in (0,\varepsilon_m]$ are subsolutions of (1.7) with the nonlinearity f_m. Since $\min_{\mathbb{R}^N} p_n > 0$, there exists $\varepsilon \in (0,\varepsilon_m]$ such that $\varepsilon\varphi_m \le p_n$ in $\mathbb{R}^N$. Hence, the maximum principle implies that

$$v(t,x) \le p_n(x) \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

where v is the solution of the Cauchy problem (1.5) with the nonlinearity f_m and initial datum $v_0 = \varepsilon\varphi_m$. But v is nondecreasing in t and converges as $t \to +\infty$ to a solution q of (1.7) with nonlinearity f_m, such that $0 < q \le p_n$ in $\mathbb{R}^N$. By minimality of p_m (from Proposition 1.1), one gets that $p_m \le q$ in $\mathbb{R}^N$, whence

$$p_m \le p_n \text{ in } \mathbb{R}^N.$$

In other words, the sequence of functions $(p_n)_{n\in\mathbb{N}}$ is nonincreasing and then converges pointwise to a periodic function $p_\infty(x)$ ranging in $[0,M]$.

Let us now show that $p_\infty = 0$ in $\mathbb{R}^N\backslash\overline{\Omega}$. By multiplying by p_n the equation (1.7) with the nonlinearity f_n, that is $-\nabla\cdot(A(x)\nabla p_n) = f_n(x,p_n)$, and by integrating by parts over the cell C_0, it follows that

$$\int_{C_0} A\nabla p_n\cdot\nabla p_n = \int_{C_0} f_n(x,p_n)\,p_n \le \int_{C_0} f_0(x,p_n)\,p_n \le M \times \max_{\mathbb{R}^N\times[0,M]} |f_0|,$$

whence the sequence $(p_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{per}(\mathbb{R}^N)$. Since it converges monotonically to p_∞, one infers that $p_\infty \in H^1_{per}(\mathbb{R}^N)$ and $p_n \to p_\infty$ as $n \to +\infty$ weakly in $H^1_{per}(\mathbb{R}^N)$ and strongly in $L^2_{per}(\mathbb{R}^N)$. For any compact set K such that $K \subset (\mathbb{R}^N \backslash \overline{\Omega}) \cap C_0$, one has

$$\begin{aligned}
-\Big(\max_{K\times[0,M]} g_n\Big)\int_K p_n^2 \le -\int_K g_n(x,p_n)\,p_n^2 &= -\int_K f_n(x,p_n)p_n \\
&= -\int_{C_0} A\nabla p_n\cdot\nabla p_n + \int_{C_0\backslash K} f_n(x,p_n)\,p_n \\
&\le \int_{C_0\backslash K} f_0(x,p_n)\,p_n \le M \times \max_{\mathbb{R}^N\times[0,M]} |f_0|.
\end{aligned}$$

The assumption (1.16) yields $\max_{K\times[0,M]} g_n \to -\infty$ as $n \to +\infty$, whence $p_\infty = 0$ a.e. in any such compact K. Finally, $p_\infty = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$. Therefore, the restriction of p_∞ on Ω is in $H^1_{0,per}(\Omega)$. Furthermore, since

$$-\nabla\cdot(A(x)\nabla p_n) = F(x,p_n) \text{ in } \overline{\Omega}, \tag{3.1}$$

the function p_∞ is a solution of the same equation in Ω in the weak $H^1_{0,per}(\Omega)$ sense. The elliptic regularity theory then implies that, up a negligible set, p_∞ is actually a $C^{2,\alpha}(\overline{\Omega})$ solution of (1.18) and the convergence $p_n \to p_\infty$ holds at least in the $C^2_{loc}(\Omega)$ sense.

Lastly, let us show that $p_\infty \ge p$ in Ω. Since p_∞ is nonnegative and $p = 0$ in all Ω_i with $i \notin I_-$, one only needs to prove that $p_n \ge p$ in Ω_i for all $i \in I_-$ and for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ and $i \in I_-$, observe that the function p_n is a supersolution of (2.3) in Ω_i, because it solves (3.1) in $\overline{\Omega_i}$ and $p_n > 0$ on $\partial\Omega_i$. Since $\min_{\overline{\Omega_i}} p_n \ge \min_{\mathbb{R}^N} p_n > 0$, there is $\varepsilon > 0$ such that $\varepsilon\widetilde{\varphi}_i \le p_n$ in $\overline{\Omega_i}$, where $\widetilde{\varphi}_i$ solves (2.4). Since $\lambda_{1,\Omega_i,D} < 0$, one can even assume without loss of generality that $\varepsilon\widetilde{\varphi}_i$ is a subsolution of (2.3), in the sense of (2.5). Therefore,

$$w(t,x) \le p_n(x) \text{ for all } (t,x) \in \mathbb{R}_+ \times \overline{\Omega_i},$$

where w denotes the solution of the Cauchy problem (2.6) in $\overline{\Omega_i}$ with initial datum $\varepsilon\widetilde{\varphi}_i$. Since w is nondecreasing in t, it converges as $t \to +\infty$ to a solution w_∞ of (2.3) such that $0 < \varepsilon\widetilde{\varphi}_i \le w_\infty \le p_n$ in Ω_i. From the construction of p in Theorem 1.2 and its minimality, one infers that $p \le w_\infty$ in $\overline{\Omega_i}$, whence

$$p \le p_n \text{ in } \overline{\Omega_i}.$$

As a conclusion, $p \le p_n$ in $\overline{\Omega}$ for all $n \in \mathbb{N}$, whence $p \le p_\infty$ in $\overline{\Omega}$. The proof of Theorem 1.3 is thereby complete. □

Remark 3.1 We first show in this remark that, if F fulfills the KPP condition (1.4) in Ω, that is if

$$s \mapsto \frac{F(x,s)}{s} \text{ is decreasing in } s > 0 \text{ for all } x \in \Omega, \tag{3.2}$$

then $p_\infty = p$ in $\overline{\Omega}$. Consider first $i \in I_-$ and let us prove that the function p solving (2.3) in $\overline{\Omega_i}$ is unique. The proof is similar to the ones used for instance in [2, 5, 7] and it is just sketched. Let q be any periodic solution of (2.3) in $\overline{\Omega_i}$. From the proof of Theorem 1.2, one knows that $q \ge p$ in $\overline{\Omega_i}$. But $\varepsilon q \le p$ in $\overline{\Omega_i}$ for $\varepsilon > 0$ small enough, from the Hopf lemma applied to p. Therefore, the quantity $\varepsilon^* = \sup\left\{\varepsilon > 0,\ \varepsilon q \leq p \text{ in } \overline{\Omega_i}\right\}$ is a positive real number. If $\varepsilon^* < 1$, then

$$-\nabla \cdot (A(x)\nabla(\varepsilon^* q)) - F(x, \varepsilon^* q) < 0 \text{ in } \Omega_i,$$

from (3.2). The strong maximum principle and Hopf lemma then imply that $\varepsilon^* q < p$ in Ω_i and even $(\varepsilon^* + \eta)q < p$ in Ω_i for all $\eta \in [0, \eta_0]$ and for some $\eta_0 > 0$. This contradicts the maximality of ε^*. Consequently, $\varepsilon^* \geq 1$, whence $q \leq p$ in $\overline{\Omega_i}$ and finally $q = p$ in $\overline{\Omega_i}$. Actually, with the same arguments as those used in the proof of Theorem 1.2, the same conclusion holds even if q is not assumed to be periodic. Now, if $i \notin I_-$, then we prove that there does not exist any solution q of (2.3) that is positive in Ω_i (or in any of its connected components). Indeed, since $F(x, s) < \zeta(x)s$ for all $x \in \Omega_i$ and $s > 0$ from (3.2), there holds

$$-\nabla \cdot (A(x)\nabla(\varepsilon\widetilde{\varphi}_i)) - F(x, \varepsilon\widetilde{\varphi}_i) > -\varepsilon\nabla \cdot (A(x)\nabla\widetilde{\varphi}_i) - \zeta(x)\varepsilon\widetilde{\varphi}_i = \lambda_{1,\Omega_i,D}\varepsilon\widetilde{\varphi}_i \geq 0 \text{ in } \Omega_i$$

for all $\varepsilon > 0$, where $\widetilde{\varphi}_i$ solves (2.4) in Ω_i, with $\lambda_{1,\Omega_i,D} \geq 0$. In other words, $\varepsilon\widetilde{\varphi}_i$ is a strict supersolution of (2.3) for all $\varepsilon > 0$. It follows with the same arguments as above or as in the proof of Theorem 1.2 that $q \leq \varepsilon\widetilde{\varphi}_i$ for all $\varepsilon > 0$, for any solution q of (2.3). Therefore, a positive periodic solution of (2.3) cannot exist, which implies that $p_\infty = p = 0$ in $\overline{\Omega_i}$ for all $i \notin I_-$. As a conclusion, the condition (3.2) implies that

$$p_\infty = p \text{ in } \overline{\Omega}. \tag{3.3}$$

On the other hand, we can construct examples for which (3.3) does not hold. It is indeed possible to construct a situation for which $\lambda_{1,D} < 0$ and there exist an index $j \in \{1, \ldots, m\}$ and $s_0 \in (0, M)$ such that $F(x, s) = \lambda s + s^2$ for all $x \in \overline{\Omega_j}$ and $s \in [0, s_0]$, where $\lambda > 0$ denotes the principal periodic eigenvalue of the operator $-\nabla \cdot (A(x)\nabla)$ in Ω_j with zero Dirichlet boundary condition on $\partial\Omega_j$. Thus, $\lambda_{1,\Omega_j,D} = 0$ and $j \notin I_-$. Let $\widetilde{\varphi}_j$ be the principal periodic eigenfunction of (2.4) in $\overline{\Omega_j}$ with $\zeta = \lambda$ in $\overline{\Omega_j}$, such that $\max_{\overline{\Omega_j}} \widetilde{\varphi}_j = 1$. For any $\varepsilon \in (0, s_0]$, there holds

$$-\nabla \cdot (A(x)\nabla(\varepsilon\widetilde{\varphi}_j)) - F(x, \varepsilon\widetilde{\varphi}_j) = -\varepsilon\nabla \cdot (A(x)\nabla\widetilde{\varphi}_j) - \lambda\varepsilon\widetilde{\varphi}_j - \varepsilon^2\widetilde{\varphi}_j^2 = -\varepsilon^2\widetilde{\varphi}_j^2 < 0 \text{ in } \Omega_j.$$

As above, it follows from the strong maximum principle that $\varepsilon\widetilde{\varphi}_j \leq p_n$ in $\overline{\Omega_j}$ for all $\varepsilon \in (0, s_0]$ and for all $n \in \mathbb{N}$. In particular, $0 < s_0\widetilde{\varphi}_j \leq p_\infty$ in Ω_j, whereas $p = 0$ in Ω_j by definition.

4 Pulsating travelling fronts and limiting minimal speed

In this section, we give the proof of Theorem 1.4. We establish the relationship between the pulsating travelling fronts for the problems (1.1) in $\mathbb{R}^N$ and (1.8) in Ω when the nonlinearity F is approximated with nonlinearities f_n which are very negative in $\mathbb{R}^N \backslash \overline{\Omega}$, in the sense of (1.16). We also prove that the minimal speeds of the fronts in $\mathbb{R}^N$ converge monotonically to a quantity which is equal to 0 in a direction e when the connected components of Ω are bounded with respect to e. We use especially some bounds for the minimal speeds, which involve some linear eigenvalue problems.

Throughout this section, we assume that $\lambda_{1,D} < 0$ and e is any given unit vector of $\mathbb{R}^N$. The functions F and f_n are assumed to fulfill (1.9) and (1.16). For each $n \in \mathbb{N}$, one has $\lambda_{1,n} < 0$ from Theorem 1.3. The functions p_n denote the minimal solutions of (1.7) with the nonlinearities f_n, given by Proposition 1.1, and the speeds $c_n^*(e) > 0$ denote the minimal speeds of pulsating fronts $\phi_n(x \cdot e - ct, x)$ connecting 0 to p_n for problems (1.1) in $\mathbb{R}^N$ with the nonlinearities f_n.

Proof of part a) of Theorem 1.4. Fix any two integers $n \leq m$ and let us show that $c_m^*(e) \leq c_n^*(e)$. First, remember that $0 < p_m \leq p_n \leq M$ (from Proposition 1.1 and Theorem 1.3) and that both functions p_m and p_n are periodic in $\mathbb{R}^N$. Let $\eta > 0$ be such that $0 < \eta < \min_{\mathbb{R}^N} p_m$ and $u_0 : \mathbb{R} \to [0, M]$ be defined by

$$u_0(x) = \begin{cases} 0 & \text{if } x \cdot e > 0, \\ \eta & \text{if } x \cdot e \leq 0. \end{cases}$$

Let v_n and v_m denote the solutions of the Cauchy problems (1.5) with initial datum u_0 and nonlinearities f_n and f_m respectively. Since $f_m \leq f_n$, the maximum principle yields

$$0 < v_m(t, x) \leq v_n(t, x) < M \ \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^N.$$

On the other hand, it follows from the results of Weinberger [35] that

$$\forall\, c < c_m^*(e), \quad \sup_{x \in \mathbb{R}^N,\, x \cdot e \leq ct} |v_m(t, x) - p_m(x)| \to 0 \text{ as } t \to +\infty,$$

while

$$\forall\, c > c_n^*(e), \quad \sup_{x \in \mathbb{R}^N,\, x \cdot e \geq ct} v_n(t, x) \to 0 \text{ as } t \to +\infty.$$

One infers that $c_m^*(e) \leq c_n^*(e)$. Consequently, the sequence $(c_n^*(e))_{n \in \mathbb{N}}$ is nonincreasing and it converges to a real number $c^*(e) \geq 0$.

From the assumptions (1.9) and (1.16) and the regularity of F and f_n, there exist a function $\overline{F} : (x, u) \mapsto \overline{F}(x, u)$ and a sequence of functions $(\overline{f}_n)_{n \in \mathbb{N}}$ such that: *i)* the function $\overline{F}$ is defined and continuous in $\overline{\Omega} \times \mathbb{R}_+$, of class $C^{0,\alpha}$ with respect to $x \in \overline{\Omega}$ locally uniformly in $u \in \mathbb{R}_+$, of class C^1 with respect to u with $\overline{\zeta} := \frac{\partial \overline{F}}{\partial u}(\cdot, 0) \in C^{0,\alpha}(\overline{\Omega})$, periodic with respect to $x \in \overline{\Omega}$ and $\overline{F}$ satisfies (1.9); *ii)* each function $\overline{f}_n$ is defined and continuous in $\mathbb{R}^N \times \mathbb{R}_+$, of class $C^{0,\alpha}$ with respect to $x \in \mathbb{R}^N$ locally uniformly in $u \in \mathbb{R}_+$, of class C^1 with respect to u with $\overline{\zeta}_n := \frac{\partial \overline{f}_n}{\partial u}(\cdot, 0) \in C^{0,\alpha}(\mathbb{R}^N)$, periodic with respect to $x \in \mathbb{R}^N$ and $\overline{f}_n$ satisfies (1.3); *iii)* the functions $\overline{f}_n$ satisfy (1.16) with $\overline{F}$ instead of F and $\overline{g}_n(x, u) = \overline{f}_n(x, u)/u$ if $u > 0$, $\overline{g}_n(x, 0) = \overline{\zeta}_n(x)$; *iv)* the function $\overline{F}$ satisfies

$$F(x, u) \leq \overline{F}(x, u) \text{ for all } (x, u) \in \overline{\Omega} \times \mathbb{R}_+$$

and $\overline{F}(x, u)/u$ is decreasing with respect to $u > 0$ for all $x \in \Omega$; *v)* the functions $\overline{f}_n$ satisfy

$$f_n(x, u) \leq \overline{f}_n(x, u) \text{ for all } (x, u) \in \mathbb{R}^N \times \mathbb{R}_+$$

and $\overline{f}_n(x,u)/u$ is decreasing with respect to $u>0$ for all $x\in\mathbb{R}^N$.

Let $\overline{\lambda}_{1,n}$ and $\overline{\lambda}_{1,D}$ be the principal periodic eigenvalues of problems (1.17) and (1.11) with coefficients $\overline{\zeta}_n$ and $\overline{\zeta}$ instead of ζ_n and ζ, respectively. Since $\overline{\zeta}_n\geq\zeta_n$ in $\mathbb{R}^N$ and $\overline{\zeta}\geq\zeta$ in $\overline{\Omega}$, there holds

$$\overline{\lambda}_{1,n}\leq\lambda_{1,n}\ \text{ and }\ \overline{\lambda}_{1,D}\leq\lambda_{1,D},$$

while $\overline{\lambda}_{1,n}<\overline{\lambda}_{1,D}$ and $\overline{\lambda}_{1,n}\to\overline{\lambda}_{1,D}$ as $n\to+\infty$ monotonically, from Theorem 1.3. In particular, $\overline{\lambda}_{1,n}<\overline{\lambda}_{1,D}<0$ for all $n\in\mathbb{N}$. Let $\overline{p}_n$ be the minimal periodic solution of (1.7) with the nonlinearity $\overline{f}_n$, given by Proposition 1.1. Actually, the function $\overline{p}_n$ is unique from property v) above and from [5], and it is such that $\overline{p}_n\geq p_n$ in $\mathbb{R}^N$ since $\overline{f}_n\geq f_n$ in $\mathbb{R}^N\times\mathbb{R}_+$, from the proof of Theorem 1.3. Let $\overline{c}^*_n(e)>0$ be the minimal speed of pulsating travelling fronts $\overline{\phi}_n(x\cdot e-ct,x)$ connecting 0 to $\overline{p}_n$ for problem (1.1) with the nonlinearity $\overline{f}_n$, that is $\overline{\phi}_n$ is periodic with respect to $x\in\mathbb{R}^N$, $0<\overline{\phi}_n(s,x)<\overline{p}_n(x)$ and $\overline{\phi}_n(-\infty,x)=\overline{p}_n(x)$, $\overline{\phi}_n(+\infty,x)=0$. As in the beginning of the proof of this theorem, there holds

$$0<c^*_n(e)\leq\overline{c}^*_n(e),\tag{4.1}$$

since $f_n\leq\overline{f}_n$. Furthermore, it follows from [6, 35] that $\overline{c}^*_n(e)$ is given by

$$\overline{c}^*_n(e)=\min_{\lambda>0}\frac{-\overline{k}_{e,\lambda,n}}{\lambda},\tag{4.2}$$

where $\overline{k}_{e,\lambda,n}$ denotes the principal periodic eigenvalue of the operator

$$\overline{\mathcal{L}}_{e,\lambda,n}:=-\nabla\cdot(A\nabla)+2\lambda\,Ae\cdot\nabla+\lambda\,\nabla\cdot(Ae)-\lambda^2\,Ae\cdot e-\overline{\zeta}_n\ \text{ in }\mathbb{R}^N.$$

Let us now show that, for every $\lambda\in\mathbb{R}$, one has $\overline{k}_{e,\lambda,n}\to\overline{k}_{e,\lambda,D}$ as $n\to+\infty$, where $\overline{k}_{e,\lambda,D}$ is the principal periodic eigenvalue of the operator

$$\overline{\mathcal{L}}_{e,\lambda,\Omega}:=-\nabla\cdot(A\nabla)+2\lambda\,Ae\cdot\nabla+\lambda\,\nabla\cdot(Ae)-\lambda^2\,Ae\cdot e-\overline{\zeta}\ \text{ in }\Omega$$

with zero Dirichlet boundary condition on $\partial\Omega$. The proof starts as in the proof of the convergence $\lambda_{1,n}\to\lambda_{1,D}$ in Theorem 1.3. First, it follows as in the proof of Theorem 1.3 that the sequence $(\overline{k}_{e,\lambda,n})_{n\in\mathbb{N}}$ is nondecreasing and that $\overline{k}_{e,\lambda,n}<\overline{k}_{e,\lambda,D}$ for all $n\in\mathbb{N}$. Let $\overline{\varphi}_n$ be a principal periodic eigenfunction of $\overline{\mathcal{L}}_{e,\lambda,n}$, that is

$$\overline{\mathcal{L}}_{e,\lambda,n}\overline{\varphi}_n=\overline{k}_{e,\lambda,n}\overline{\varphi}_n\text{ and }\overline{\varphi}_n>0\text{ in }\mathbb{R}^N.$$

Up to normalization, one can assume that $\|\overline{\varphi}_n\|_{L^2(C_0)}=1$. By multiplying the above equation by $\overline{\varphi}_n$, by integrating by parts over C_0 and by using Young's inequality, it follows that the sequence $(\overline{\varphi}_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{per}(\mathbb{R}^N)$. Up to extraction of a subsequence, it converges weakly in $H^1_{per}(\mathbb{R}^N)$ and strongly in $L^2_{per}(\mathbb{R}^N)$ to a nonnegative function $\overline{\varphi}_\infty\in H^1_{per}(\mathbb{R}^N)$ such that $\|\overline{\varphi}_\infty\|_{L^2(C_0)}=1$. Furthermore, since the sequence $(\overline{k}_{e,\lambda,n})_{n\in\mathbb{N}}$ is bounded and $\overline{\zeta}_n\to-\infty$ as $n\to+\infty$ locally uniformly in $\mathbb{R}^N\backslash\overline{\Omega}$, one infers as in the proof of Theorem 1.3 that $\overline{\varphi}_\infty=0$ a.e. in $\mathbb{R}^N\backslash\Omega$. The restriction of $\overline{\varphi}_\infty$ to $\overline{\Omega}$ is then a $C^{2,\alpha}(\overline{\Omega})$ periodic function such that

$$\overline{\mathcal{L}}_{e,\lambda,\Omega}\overline{\varphi}_\infty=\overline{k}_{e,\lambda,\infty}\overline{\varphi}_\infty\text{ in }\overline{\Omega}\text{ with }\overline{\varphi}_\infty=0\text{ on }\partial\Omega,$$

where $\lim_{n\to+\infty} \overline{k}_{e,\lambda,n} = \overline{k}_{e,\lambda,\infty} \le \overline{k}_{e,\lambda,D}$. Since the function $\overline{\varphi}_\infty$ is periodic, nonnegative and nontrivial, it follows that it is positive in Ω_i for some $i \in \{1,\dots,m\}$, that is $\overline{k}_{e,\lambda,\infty}$ is equal to the principal periodic eigenvalue $\overline{k}_{e,\lambda,\Omega_i,D}$ of the operator $\overline{\mathcal{L}}_{e,\lambda,\Omega}$ in Ω_i with zero Dirichlet boundary condition on $\partial\Omega_i$. But since $\overline{k}_{e,\lambda,\Omega_i,D} \ge \overline{k}_{e,\lambda,D}\,(\ge \overline{k}_{e,\lambda,\infty})$, one concludes eventually that $\overline{k}_{e,\lambda,\infty} = \overline{k}_{e,\lambda,D}$, that is

$$\overline{k}_{e,\lambda,n} \to \overline{k}_{e,\lambda,D} \text{ as } n \to +\infty. \tag{4.3}$$

Assume now that all connected components of Ω are bounded in the direction e, in the sense of (1.19). Let us show that $c^*(e) = 0$. First, it follows from (4.1), (4.2) and (4.3) that

$$0 \le c^*(e) \le \inf_{\lambda>0} \frac{-\overline{k}_{e,\lambda,D}}{\lambda}. \tag{4.4}$$

On the other hand, for every $\lambda > 0$, there is an index $i \in \{1,\dots,m\}$, which may depend on λ, such that $\overline{k}_{e,\lambda,D} = \overline{k}_{e,\lambda,\Omega_i,D}$ and thus there is a periodic function φ defined in $\overline{\Omega_i}$ such that $\overline{\mathcal{L}}_{e,\lambda,\Omega}\varphi = \overline{k}_{e,\lambda,D}\varphi$ in $\overline{\Omega_i}$ with $\varphi > 0$ in Ω_i and $\varphi = 0$ on $\partial\Omega_i$. The function $\psi = e^{-\lambda(x\cdot e)}\varphi$ satisfies

$$-\nabla\cdot(A(x)\nabla\psi) - \overline{\zeta}(x)\psi = \overline{k}_{e,\lambda,D}\psi \ \text{ in } \overline{\Omega_i} \tag{4.5}$$

with $\psi > 0$ in Ω_i and $\psi = 0$ on $\partial\Omega_i$. Let $\mathcal{C}$ be any connected component of Ω_i, that is $\mathcal{C} = \omega_i + k$ for some $k \in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$. The function ψ is positive and bounded in $\mathcal{C}$ because of (1.19) and since φ is bounded. It follows then from Hopf lemma and the smoothness of $\partial\mathcal{C}$ that there exist $r > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $B(x_n, r) \subset C$ for all $n \in \mathbb{N}$ and $\psi(x_n) \to \sup_{\mathcal{C}}\psi$ as $n \to +\infty$. By using the standard elliptic estimates and passing to the limit in (4.5) in $B(x_n, r)$, up to extraction of a subsequence, one infers that $\overline{k}_{e,\lambda,D} \ge \liminf_{n\to+\infty} -\overline{\zeta}(x_n) \ge -\max_{\overline{\Omega_i}}\overline{\zeta}$. Finally, $\overline{k}_{e,\lambda,D} \ge -\max_{\overline{\Omega}}\overline{\zeta}$ for all $\lambda > 0$, whence $c^*(e) = 0$ from (4.4).

Proof of part b) of Theorem 1.4. Let c be any positive real number such that $c \ge c^*(e)$ and let $(c_n)_{n\in\mathbb{N}}$ be any sequence such that $c_n \to c$ as $n \to +\infty$ and $c_n \ge c_n^*(e)$ for all $n \in \mathbb{N}$. Let

$$u_n(t,x) = \phi_n(x\cdot e - c_n t, x)$$

be pulsating travelling fronts for (1.1) in $\mathbb{R}^N$ with nonlinearity f_n, such that

$$0 = \phi_n(+\infty,x) < \phi_n(s,x) < \phi_n(-\infty,x) = p_n(x) \le M \ \text{ for all } (s,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Actually, from [16], each solution u_n satisfies $(u_n)_t > 0$ in $\mathbb{R}\times\mathbb{R}^N$.

On the one hand, since $0 < u_n(t,x) < p_n(x)$ in $\mathbb{R}\times\mathbb{R}^N$, Theorem 1.3 implies that $u_n \to 0$ in $L^1_{loc}(\mathbb{R}\times(\mathbb{R}^N\backslash\Omega))$. On the other hand, since $f_n(x,s) = F(x,s)$ for all $(x,s) \in \overline{\Omega}\times\mathbb{R}_+$, it follows from standard parabolic estimates that there exists a function $u : \mathbb{R}\times\Omega \to [0,M]$ such that, up to extraction of a subsequence, $u_n \to u$ as $n \to +\infty$ in C^1_t and C^2_x in $\mathbb{R}\times\Omega$, where u obeys

$$u_t - \nabla\cdot(A(x)\nabla u) = F(x,u) \ \text{ in } \mathbb{R}\times\Omega$$

and $0 \le u(t,x) \le p_\infty(x) \le M$ for all $(t,x) \in \mathbb{R}\times\Omega$, under the notation of Theorem 1.3. In particular, the function u can be extended continuously by 0 on $\mathbb{R}\times\partial\Omega$ and, from parabolic regularity, the function u is a classical solution of (1.8) in $\mathbb{R}\times\overline{\Omega}$ (of course, one could also extend u by 0 in $\mathbb{R}\times(\mathbb{R}^N\backslash\Omega)$ and u would then be continuous in $\mathbb{R}\times\mathbb{R}^N$). Moreover, the equalities

$$u_n\Big(t+\frac{k\cdot e}{c_n},x\Big)=u_n(t,x-k)\text{ in }\mathbb{R}\times\mathbb{R}^N$$

for all $k\in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$ carry over at the limit, whence $u(t+(k\cdot e)/c,x)=u(t,x-k)$ in $\mathbb{R}\times\overline{\Omega}$ for all $k\in L_1\mathbb{Z}\times\cdots\times L_N\mathbb{Z}$. In other words, the function u can be written as $u(t,x)=\phi(x\cdot e-ct,x)$ in $\mathbb{R}\times\overline{\Omega}$ where $\phi:\mathbb{R}\times\overline{\Omega}\to[0,M]$ is such that $\phi(s,\cdot)$ is periodic in $\overline{\Omega}$ for all $s\in\mathbb{R}$. Lastly, since all functions u_n are increasing in time in $\mathbb{R}\times\mathbb{R}^N$, the function u is such that $u_t\ge 0$ in $\mathbb{R}\times\overline{\Omega}$. From the previous observations and parabolic regularity, there are then two periodic functions $u^{\pm}$ defined in $\overline{\Omega}$ such that $0\le u^-\le u^+\le p_\infty$ in $\overline{\Omega}$, $u(t,x)\to u^{\pm}(x)$ as $t\to\pm\infty$ in $C^2_{loc}(\overline{\Omega})$ and $u^{\pm}$ obey

$$\left\{\begin{array}{rcll}-\nabla\cdot(A(x)\nabla u^{\pm}) & = & F(x,u^{\pm}) & \text{in }\overline{\Omega},\\ u^{\pm} & = & 0 & \text{on }\partial\Omega.\end{array}\right.$$

Let now any index $i\in I_-$, that is $\lambda_{1,\Omega_i,D}<0$ in the sense of (1.12). From the proof of Theorem 1.2, there is a minimal periodic solution $\widetilde{p}_i$ of (2.3). Furthermore, in $\overline{\Omega_i}$, there holds $\widetilde{p}_i=p\le p_\infty\le p_n$ for all $n\in\mathbb{N}$, under the notation of Theorem 1.3. Therefore, one can always shift in time the functions u_n so that, say,

$$\int_{C_0\cap\Omega_i}u_n(0,x)\,dx=\frac{1}{2}\int_{C_0\cap\Omega_i}\widetilde{p}_i(x)\,dx,$$

where we recall that $C_0=[0,L_1]\times\cdots\times[0,L_N]$. From Lebesgue's dominated convergence theorem, the function u satisfies the same equality at the limit, whence

$$0\le\int_{C_0\cap\Omega_i}u^-(x)\,dx\le\frac{1}{2}\int_{C_0\cap\Omega_i}\widetilde{p}_i(x)\,dx\le\int_{C_0\cap\Omega_i}u^+(x)\,dx$$

by monotonicity of u with respect to t. The minimality of $\widetilde{p}_i$ and the strong maximum principle imply that $u^-=0$ in $\overline{\Omega_i}$, while $u^+>0$ in Ω_i, again from the strong maximum principle. If we further assume that F satisfies the KPP assumption (3.2) in Ω (or just in Ω_i), then it follows from Remark 3.1 that the solution of (2.3) is actually unique, whence $u^+=\widetilde{p}_i$ in $\overline{\Omega_i}$ in this case.

Proof of part c) of Theorem 1.4. Firstly, it follows from [3, 35] that, for each $n\in\mathbb{N}$,

$$c^*_n(e)\ge\min_{\lambda>0}\frac{-k_{e,\lambda,n}}{\lambda}=\frac{-k_{e,\lambda_n,n}}{\lambda_n}$$

for some $\lambda_n>0$, where $k_{e,\lambda,n}$ denotes the principal periodic eigenvalue of the operator

$$\mathcal{L}_{e,\lambda,n}:=-\nabla\cdot(A\nabla)+2\lambda\,Ae\cdot\nabla+\lambda\,\nabla\cdot(Ae)-\lambda^2\,Ae\cdot e-\zeta_n\ \text{ in }\mathbb{R}^N. \tag{4.6}$$

Since, as above, $k_{e,\lambda,n}\to k_{e,\lambda,D}$ as $n\to+\infty$ nondecreasingly for every $\lambda\in\mathbb{R}$, where $k_{e,\lambda,D}$ is the principal periodic eigenvalue of the operator

$$\mathcal{L}_{e,\lambda,\Omega}:=-\nabla\cdot(A\nabla)+2\lambda\,Ae\cdot\nabla+\lambda\,\nabla\cdot(Ae)-\lambda^2\,Ae\cdot e-\zeta\ \text{ in }\Omega$$

with zero Dirichlet boundary condition on $\partial\Omega$, it follows that

$$c^*_n(e) \geq \frac{-k_{e,\lambda_n,n}}{\lambda_n} \geq \frac{-k_{e,\lambda_n,D}}{\lambda_n} \geq \inf_{\lambda>0} \frac{-k_{e,\lambda,D}}{\lambda}$$

for all $n \in \mathbb{N}$, whence

$$c^*(e) \geq \inf_{\lambda>0} \frac{-k_{e,\lambda,D}}{\lambda}. \tag{4.7}$$

Furthermore, the maps $\lambda \mapsto -k_{e,\lambda,n}$ are all convex and their derivatives at $\lambda = 0$ are all equal to 0, see [3, 6]. In particular, for every $n \in \mathbb{N}$, $-k_{e,\lambda,n}$ is nondecreasing with respect to $\lambda \geq 0$ and $-k_{e,\lambda,n} \geq -k_{e,0,n} = -\lambda_{1,n}$ for all $\lambda \in \mathbb{R}$. By passing to the limit as $n \to +\infty$ pointwise in λ, one gets that the map $\lambda \mapsto -k_{e,\lambda,D}$ is convex in $\mathbb{R}$, nondecreasing in $\mathbb{R}_+$, and there holds $-k_{e,\lambda,D} \geq -k_{e,0,D} = -\lambda_{1,D} > 0$ for all $\lambda \in \mathbb{R}$. Notice here that, if assumption (1.19) is made, then $-k_{e,\lambda,D} \leq \max_{\overline{\Omega}} \zeta$ for all λ, under the notation used in the proof of part a). Therefore, the infimum in (4.7) is not reached in general.

Because of (4.7), the inequality $-k_{e,\lambda,D} \geq -\lambda_{1,D} > 0$ and the limit $\lim_{n\to+\infty} k_{e,\lambda,n} = k_{e,\lambda,D}$ for all λ, it follows that, in order to show the positivity of $c^*(e)$, it is sufficient to prove that there exist $\Lambda > 0$ and $\alpha > 0$ such that

$$-k_{e,\lambda,n} \geq \alpha\, \lambda^2 \text{ for all } \lambda \geq \Lambda \text{ and for all } n \in \mathbb{N}. \tag{4.8}$$

Of course, from the proof of part a), this cannot be always true. However, assuming from now on that A is constant, we shall now show that (4.8) holds under conditions (1.20) or (1.21). Assume first that there exist a unit vector $e' \neq \pm e$ and two real numbers $a < b$ such that (1.20) is fulfilled, that is

$$\Omega \supset S_{e',a,b} := \big\{x \in \mathbb{R}^N,\ a < x\cdot e' < b\big\}.$$

For any $\lambda > 0$, let ψ_λ be the function defined in $\overline{S_{e',a,b}}$ by

$$\psi_\lambda(x) = e^{\lambda'(x\cdot e')} \cos\Big(\frac{\pi}{b-a} \times \big(x\cdot e' - \frac{a+b}{2}\big)\Big),$$

where $\lambda' = \lambda\,(Ae\cdot e')/(Ae'\cdot e')$. The function ψ_λ is bounded and of class $C^\infty\big(\overline{S_{e',a,b}}\big)$, it is positive in $S_{e',a,b}$ and vanishes on $\partial S_{e',a,b}$. Furthermore, since $\zeta_n = \zeta$ in $\overline{\Omega} \supset \overline{S_{e',a,b}}$, it is straightforward to check that

$$\mathcal{L}_{e,\lambda,n}\psi_\lambda = \Big(\frac{\pi^2(Ae'\cdot e')}{(b-a)^2} - \zeta(x) - 2\,\alpha\,\lambda^2\Big)\psi_\lambda \text{ in } \overline{S_{e',a,b}}$$

for all $n \in \mathbb{N}$, where $\alpha = (Ae\cdot e)/2 - (Ae\cdot e')^2/(2\,Ae'\cdot e') > 0$ from Cauchy-Schwarz inequality, since the unit vectors e and e' are not parallel. Since ζ is bounded in $\overline{\Omega}$, it follows that there exists $\Lambda > 0$ such that $\mathcal{L}_{e,\lambda,n}\psi_\lambda \leq -\alpha\,\lambda^2\,\psi_\lambda$ in $\overline{S_{e',a,b}}$ for all $\lambda \geq \Lambda$ and $n \in \mathbb{N}$. This inequality yields (4.8), as in the course of the proof of Proposition 1.1. We just sketch the proof here. Fix any $\lambda \geq \Lambda$ and $n \in \mathbb{N}$ and let ϕ_n be a principal periodic eigenfunction of the operator $\mathcal{L}_{e,\lambda,n}$. Namely, $\mathcal{L}_{e,\lambda,n}\phi_n = k_{e,\lambda,n}\phi_n$ and ϕ_n is periodic and positive in $\mathbb{R}^N$. Define

$$\varepsilon^* = \sup\big\{\varepsilon > 0,\ \varepsilon\psi_\lambda \leq \phi_n \text{ in } \overline{S_{e',a,b}}\big\}.$$

Owing to the definition of ψ_λ and the uniform positivity of ϕ_n, the quantity ε^* is a positive real number. Furthermore, $\varepsilon^*\psi_\lambda \le \phi_n$ in $\overline{S_{e',a,b}}$ and there is a sequence $(x_m)_{m\in\mathbb{N}}$ of points in $S_{e',a,b}$ such that $\liminf_{m\to+\infty} d(x_m,\partial S_{e',a,b})>0$, $\lim_{m\to+\infty}(\varepsilon^*\psi_\lambda(x_m)-\phi_n(x_m))=0$ and $\liminf_{m\to+\infty}\mathcal{L}_{e,\lambda,n}(\varepsilon^*\psi_\lambda-\phi_n)(x_m)\ge 0$. Since there holds $\mathcal{L}_{e,\lambda,n}\phi_n(x_m)=k_{e,\lambda,n}\phi_n(x_m)$ and $\mathcal{L}_{e,\lambda,n}\psi_\lambda(x_m)\le -\alpha\lambda^2\psi_\lambda(x_m)$ for every $m\in\mathbb{N}$, one concludes that $k_{e,\lambda,n}\le -\alpha\lambda^2$, that is (4.8). This yields the desired inequality $c^*(e)>0$, as already emphasized.

Assume now that there exist a unit vector e', a point $x_0\in\mathbb{R}^N$ and a positive real number r such that e' is an eigenvector of A with $e'\cdot e\neq 0$ and (1.21) holds. Let $\beta>0$ be such that $Ae'=\beta e'$. Since the matrix A is symmetric, there is an orthonormal family of eigenvectors $e'_1,\dots,e'_{N-1}$ of A in $\mathbb{R}^N$ such that $e'_i\cdot e'=0$ for all $1\le i\le N-1$. Even if it means decreasing $r>0$ in (1.21), one can assume without loss of generality that

$$\Omega\supset C_{e',r}:=\big\{x\in\mathbb{R}^N,\ |(x-x_0)\cdot e'_i|<r \text{ for all } 1\le i\le N-1\big\}.$$

For any $\lambda>0$, let ψ_λ be the function defined in $\overline{C_{e',r}}$ by

$$\psi_\lambda(x)=\prod_{1\le i\le N-1} e^{\lambda'_i(x\cdot e'_i)}\cos\Big(\frac{\pi(x-x_0)\cdot e'_i)}{2\,r}\Big),$$

where $\lambda'_i=\lambda\,(Ae\cdot e'_i)/(Ae'_i\cdot e'_i)$. The function ψ_λ is bounded and of class $C^\infty\big(\overline{C_{e',r}}\big)$, it is positive in $C_{e',r}$ and vanishes on $\partial C_{e',r}$. Furthermore, since $\zeta_n=\zeta$ in $\overline{\Omega}\supset\overline{C_{e',r}}$, it is straightforward to check that

$$\mathcal{L}_{e,\lambda,n}\psi_\lambda=\Big[\Big(\sum_{1\le i\le N-1}\frac{\pi^2(Ae'_i\cdot e'_i)}{4r^2}\Big)-\zeta(x)-2\,\alpha\,\lambda^2\Big]\psi_\lambda\ \text{ in } \overline{C_{e',r}}$$

for all $n\in\mathbb{N}$, where $\alpha=\beta(e\cdot e')^2/2>0$ since $\beta>0$ and $e\cdot e'\neq 0$ by assumption. Thus, one concludes as above that there is $\Lambda>0$ such that $\mathcal{L}_{e,\lambda,n}\psi_\lambda\le -\alpha\,\lambda^2\psi_\lambda$ in $\overline{C_{e',r}}$ for all $\lambda\ge\Lambda$ and $n\in\mathbb{N}$. This yields (4.8) and finally $c^*(e)>0$. The proof of Theorem 1.4 is thereby complete. □

Remark 4.1 In the case when the functions f_n fulfill the KPP assumption (1.4), then $c^*(e)$ is given by an explicit variational formula. Namely, under assumption (1.4) for the functions f_n, it follows from the proof of part a) of Theorem 1.4 with the choices $\overline{f}_n=f_n$ and $\overline{F}=F$ that $c^*(e)\le\inf_{\lambda>0}-k_{e,\lambda,D}/\lambda$, because of (4.4). On the other hand, the reverse inequality (4.7) always holds, from the proof of part c) of Theorem 1.4. As a conclusion, the assumption (1.4) for the functions f_n yields

$$c^*(e)=\inf_{\lambda>0}\frac{-k_{e,\lambda,D}}{\lambda}.$$

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